AN ABSTRACT THEORY FOR THE DOMAIN REDUCTION METHOD*

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Abstract. The domain reduction method uses a finite group of symmetries of a system of linear equations arising by discretization of partial differential equations to obtain a decomposition into independent subproblems, which can be solved in parallel. This paper develops a theory for this class of methods based on known results from group representation theory and algebras of finite groups. The main theoretical result is that if the problem splits into subproblems based on isomorphic subdomains, then the group of symmetries must be commutative. General decompositions are then obtained by nesting decompositions based on commutative groups of symmetries.

Key words. Parallel Computing, Group Representation, Group Algebras, Symmetries, Partial Differential Equations, Boundary Value Problems

1. Introduction. This paper is concerned with the problem of decomposing a system of linear algebraic equations into subsystems based on symmetries of the operator and the solution space. The subproblems can then be solved independently and in parallel. The system to be solved is associated in a natural way with a domain in an Euclidean space, such as in the case of discretizations of partial differential equations on symmetrical domains. The group of symmetries of the domain then induces a group of transformations in functional spaces on that domain. If the operator of the system to be solved satisfies the same symmetries, then the symmetries can be used to develop a decomposition.

Methods of this type were proposed by Douglas and Miranker [7] and further studied by Douglas and Smith [9] essentially for symmetries of the square along the axis parallel with the sides, giving decomposition into four subproblems defined on smaller identical squares. Each of the subproblems on the squares can be further decomposed into two equal sized, smaller problems [3], giving a decomposition into eight subproblems defined on subdomains of the same area but not the same shape. Douglas [4] studied when to use a simple or a high way domain reduction, for serial, distributed, or parallel computing.

When applying this method to parallel computing, the bottleneck is the size of the biggest concurrent subproblem to be solved, and load balancing is of great importance.

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Thus, even if some of those eight subproblems can be decomposed further, the fact that at least one of them cannot be decomposed limits the parallel efficiency. Douglas and Mandel [6] gave a decomposition for a cube into 60 or 64 subproblems using a similar nested decomposition techniques as in [3], and also reported on computational experience with parallel implementations.

The present paper gives an abstract systematic framework for the methods from [3, 4, 6, 9] using group theoretic tools, thus eliminating the difficulty that the decomposition for more complicated groups was previously found only on a case-by-case basis. A different group theoretic approach was first presented by Allgower, Böhmer, and Mei [1] and further developed by Allgower, Böhmer, Georg, and Miranda [2]. But the theory in [1, 2] still leaves open the question how to construct the decomposition in the general case, cf., §8 below.

The basic idea of the domain reduction method is to split the solution space into invariant subspaces of a group of symmetries. This paper is based on the observation that such decompositions are studied in the group representation theory and their properties are well known [12]. In particular, the decomposition can be shown to be uniquely defined by the group under suitable assumptions. The projection operators are linear combinations of the symmetries of the solution space, which lends itself to analysis in the framework of group algebras, which consist of all such linear combinations.

Allgower, Böhmer, and Mei [1] arrived independently at a decomposition for the square into six subproblems, two of them of twice the size of the remaining four. Their decomposition is based on the symmetry group of the square, which consists of eight transformations. Their approach is based on decomposition the solution space into a direct sum of fixed point subspaces for certain subgroups of their group of transformations $\mathbf{D}_4 \times \mathbf{Z}_2$, where $\mathbf{D}_4$ is the eight element symmetry group of the square and $\mathbf{Z}_2$ is the group of multiplications by ± 1. This method is based on approaches known in bifurcation theory (see [10] and [11]). Unfortunately, it is not clear how to select these subgroups in the general case. However, their subspaces are invariant subspaces of the transformation group $\mathbf{D}_4$, and they can be described using the present framework, cf., §8 of this paper. The results of this paper show that it is not possible to use the symmetry group $\mathbf{D}_4$ to obtain a decomposition into eight subspaces, because the group $\mathbf{D}_4$ is not commutative. The approach of [1] was extended to general groups by Allgower, Böhmer, Georg, and Miranda [2]. A related approach was used by Munthe-Kaas [13] in developing Fast Fourier Transforms exploiting group symmetries; it should be noted that the method of this paper can be also interpreted as a block-wise Fourier transform on the group, with the components in the subspaces of the decomposition playing the role of Fourier components.

For a simple introduction to the group representation theory and classical applications of group representation in other areas of applied mathematics, see [15].

The theory presented in this paper starts as the search for a family of projections on a linear space V that can be given as linear combinations of transformations from a given finite group $\mathcal{G}$ of linear operators, and such that V splits into the direct sum of subspaces V_i, given as the ranges of the projections (§2). This formulation naturally leads to the concept of group algebras. Pertinent properties of group algebras are then surveyed in §3 and it is shown that construction of those projections is equivalent to a well-known problem from group representation theory, namely finding an idempotent basis of the group algebra. The symmetry group of the domain defines its subdomains, called fundamental domains, which are mapped onto each other by the transformations in the group (§4). In §5, conditions are considered when it is possible to reduce the subproblems to one or more fundamental domains. It is shown that if every subproblem can be reduced to one fundamental domain, then the group of transformations must be commutative. Since it is important for parallel implementations that all subproblems should be of about the same size to achieve good load balancing, we then restrict our attention to commutative groups and the general construction of the projections in that case is investigated in §6. Since commutativity of the symmetry group rather restricts possible applications, nested decompositions are used to construct general decompositions in §7. Such decompositions, however, cannot often be obtained directly from mappings of the domain. The present theory is compared with that of [1] and [2] in §8. §9 discusses the role of boundary conditions. The paper is concluded by several examples in §10.

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2. Domain Reduction. We are interested in the solution of the linear problem

$$Lu = f \tag{2.1}$$

on a linear space V for a given linear mapping $L : V \to V$. A group $\mathcal{G}$ of linear mappings on V is also given such that L commutes with the mappings in the group $\mathcal{G}$,

$$LG = GL, \qquad \forall G \in \mathcal{G}. \tag{2.2}$$

We wish to use the group $\mathcal{G}$ to define a decomposition of the space V into a direct sum

$$V = V_1 \oplus \cdots \oplus V_m, \tag{2.3}$$

where the associated projections Π_i,

$$I = \Pi_1 + \cdots + \Pi_m, \qquad V_i = \operatorname{Range} \Pi_i, \tag{2.4}$$

$$\Pi_i \Pi_j = 0, \qquad i \neq j, \tag{2.5}$$

are linear combinations of the elements of the group $\mathcal{G}$,

$$\Pi_i = \sum_{G \in \mathcal{G}} c_i(G) G. \tag{2.6}$$

It follows from (2.2) and (2.6) that L and the projections Π_i commute,

$$L\Pi_i = L \sum_{G\in\mathcal{G}} c_i(G)G = \sum_{G\in\mathcal{G}} c_i(G)LG = \sum_{G\in\mathcal{G}} c_i(G)GL = \Pi_i L.$$

So, for any $v_i \in V_i$,

$$Lv_i = L\Pi_i v_i = \Pi_i L v_i \in V_i,$$

which means that each $LV_i \subseteq V_i$. Thus, (2.1) splits into m independent subproblems,

$$Lu_i = f_i \equiv \Pi_i f, \qquad u_i \in V_i. \tag{2.7}$$

To obtain a computationally desirable method in the discrete case, the subproblems (2.7) are expressed as matrix problems on a space of a lower dimension than V_i. For that purpose, *prolongations*

$$P_i : W_i \to V_i,$$

are needed that are of full rank (otherwise, the operator L_i below would be singular). Then

$$\Pi_i = P_i(P_i^T P_i)^{-1} P_i^T$$

and (2.7) can be solved as

$$L_i w_i = g_i, \qquad w_i \in W_i, \tag{2.8}$$

where

$$L_i = P_i^T L P_i \qquad \text{and} \qquad g_i = P_i^T f. \tag{2.9}$$

Hence, u_i is obtained from $u_i = P_i w_i$.

Since the dimensions of the spaces W_i are smaller than those of V_i, the subproblems (2.8) are cheaper to solve than the original problem (2.1). In our application, the dimension of V is about the sum of the dimensions of W_i (except perhaps for degrees of freedom that correspond to boundaries of physical domains), so if all the dimensions of W_i are about the same, almost perfect speedup can be obtained by solving them in parallel. As it was observed in [2], very significant savings can be accomplished even on a serial computer, since the cost of direct solution of a linear system grows as the second of third power of the dimension (depending on the sparsity of the system matrix).

Note that for some problems the following generalization of this approach may be used with a different decomposition (2.3) or space of right hand sides. Let $L : V \to \tilde{V}$ (possibly $V = \tilde{V}$). Instead of L satisfying (2.2), let it satisfy the condition

$$LG = \chi(G)L, \tag{2.10}$$

where $\chi : \mathcal{G} \to \tilde{\mathcal{G}}$ is a homomorphism of $\mathcal{G}$ onto a group $\tilde{\mathcal{G}}$ of linear mappings on $\tilde{V}$.

Then the decomposition (2.4) in V defines a similar decomposition in $\tilde{V}$. This follows immediately from the fact that χ is a group homomorphism that if $\Pi_i = \sum_{G\in\mathcal{G}} c_i(G)G$ are projections, then $\tilde{\Pi}_i = \sum_{G\in\mathcal{G}} c_i(G)\chi(G)$ are also projections. Hence, on $\tilde{V}$,

$$I = \tilde{\Pi}_1 + \cdots + \tilde{\Pi}_m, \qquad \tilde{\Pi}_i\tilde{\Pi}_j = 0, \qquad i \neq j.$$

Further, (2.10) gives that $L\Pi_i = \tilde{\Pi}_i L$. So,

$$LV_i \subseteq \tilde{V}_i,$$

as in the earlier case.

3. Group Algebra. The projections given by (2.6) can be studied using group algebras. The basic results that are pertinent to this paper from the theories of groups and group algebras are reviewed, following [12].

For a finite group $\mathcal{G}$, the associated *group algebra* $\mathcal{A}$ is the set of formal sums

$$\sum_{G\in\mathcal{G}} c(G)G \tag{3.1}$$

where $c(G)$ are scalars. $\mathcal{A}$ is equipped with the structure of a linear space and vector multiplication,

$$\begin{aligned} \Big(\sum_{G\in\mathcal{G}} c(G)G\Big)\Big(\sum_{H\in\mathcal{G}} d(H)H\Big) &= \sum_{G\in\mathcal{G}}\sum_{H\in\mathcal{G}} c(G)d(H)GH \\ &= \sum_{F\in\mathcal{G}}\Big(\sum_{H\in\mathcal{G}} c(FH^{-1})d(H)\Big)F, \end{aligned} \tag{3.2}$$

where the substitution $F = GH$ and rearrangement were used [12, p. 86].

We shall assume that

$$\textit{the group } \mathcal{G} \textit{ is linearly independent} \tag{3.3}$$

as a set of linear operators on V. Then $\mathcal{A}$ is isomorphic to the algebra of linear operators on V generated by all $G \in \mathcal{G}$, that is, (3.1) can be understood to mean scalar multiplication and summation of linear operators on V. Similarly, (3.2) means the composition of linear operators on V.

A subspace $\mathcal{J} \subseteq \mathcal{A}$ is called a *right ideal* of $\mathcal{A}$ if

$$ab \in \mathcal{J}, \qquad \forall a \in \mathcal{J}, b \in \mathcal{A}.$$

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$\mathcal{J}$ is *two-sided ideal* of $\mathcal{A}$ if it is both a left and a right ideal.

Every $G \in \mathcal{G}$ defines linear mappings on $\mathcal{A}$.

$$G_L : a \mapsto Ga, \qquad G_R : a \mapsto aG,$$

are called *left* and *right multiplication*, respectively.

It is easy to see that *a left ideal $\mathcal{J}$ of $\mathcal{A}$ is an invariant subspace of the group $\mathcal{G}$,*

$$G\mathcal{J} \subseteq \mathcal{J}, \qquad \forall G \in \mathcal{G},$$

and any invariant subspace of $\mathcal{G}$ is a left ideal of $\mathcal{A}$ [12, p. 91].

If $\mathcal{J}$ is a left ideal, then every $G \in \mathcal{G}$ defines a linear mapping of $\mathcal{J}$ into itself by left multiplication restricted to $\mathcal{J}$, which we denote by $G_L|_{\mathcal{J}}$. The mapping

$$G \in \mathcal{G} \mapsto G_L|_{\mathcal{J}} \tag{3.4}$$

is a group homomorphism; it is called the *representation* of $\mathcal{G}$ on the left ideal $\mathcal{J}$. Two left ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ are *equivalent* if the corresponding representations are equivalent in the sense that there exists an invertible linear mapping $A : \mathcal{J}_1 \to \mathcal{J}_2$ such that

$$Ga = AGA^{-1}a, \qquad \forall a \in \mathcal{J}_2,\ G \in \mathcal{G}, \tag{3.5}$$

(cf., [12, pp. 66, 96]). In particular, note that two equivalent ideals have the same dimension and that the images of the representations (3.4) are isomorphic groups.

A *minimal ideal* is an ideal which does not contain any nontrivial ideal of the same type. Thus, a minimal left ideal contains no left ideals except the trivial ideal $\{0\}$ and itself.

The following theorem applied recursively shows that *any left ideal is a direct sum of minimal left ideals.*

Theorem 3.1. (*Maschke's Theorem* [12, p. 73]) *Let $\mathcal{J}$ and $\mathcal{J}'$ be left ideals of $\mathcal{A}$, and $\mathcal{J}' \subset \mathcal{J}$. Then there exists a left ideal $\mathcal{J}''$ such that*

$$\mathcal{J} = \mathcal{J}' \oplus \mathcal{J}''.$$

Thus, the algebra itself can be decomposed into minimal left ideals:

Theorem 3.2. [12, p. 92] *The group algebra $\mathcal{A}$ of $\mathcal{G}$ is the direct sum of minimal left ideals:*

$$\mathcal{A} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_m.$$

If

$$\mathcal{A} = \mathcal{J}'_1 \oplus \cdots \oplus \mathcal{J}'_{m'}$$

is another decomposition of $\mathcal{A}$ into minimal left ideals, then $m = m'$ and after renumbering, the ideals $\mathcal{J}_i$ and $\mathcal{J}'_i$ are equivalent, $i = 1, \cdots, m$.

An element $\Pi \in \mathcal{A}$ is called *idempotent* if $\Pi^2 = \Pi$. Note that when Π is understood as a linear operator on V, it is then a projection onto a subspace of V. For any element $b \in \mathcal{A}$,

$$\mathcal{A}b = \{ab : a \in \mathcal{A}\}$$

is a left ideal that is called the *principal ideal generated* by b.

THEOREM 3.3. [12, p. 97] *Let $\mathcal{A}$ be the direct sum of nontrivial left ideals,*

$$\mathcal{A} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_m. \tag{3.6}$$

Then there exist unique idempotents $\Pi_1, \cdots, \Pi_m$ which form a decomposition of unity,

$$\Pi_1 + \cdots + \Pi_m = I \text{ on } \mathcal{A}, \tag{3.7}$$

and each Π_i generates the corresponding ideal $\mathcal{J}_i$. In addition,

$$\Pi_i \Pi_j = 0, \qquad i \neq j. \tag{3.8}$$

Theorem 3.3 shows that to find a decomposition of unity satisfying (3.7) and (3.8), it is enough to to find the corresponding ideals of $\mathcal{A}$. Theorem 3.4 shows this is also necessary.

THEOREM 3.4. *Let*

$$V = V_1 \oplus \cdots \oplus V_m$$

and assume that the corresponding projections $\Pi_i : V \to V_i$ have the form (2.6). Then $\mathcal{J}_i = \mathcal{A}\Pi_i$ are left ideals of the group algebra $\mathcal{A}$, and (3.6)–(3.8) hold.

Proof. From the properties of a direct sum, (3.7) and (3.8) are immediate. It is obvious that $\mathcal{J}_i = \mathcal{A}\Pi_i$ are left ideals. It only remains to prove (3.6). Any $a \in \mathcal{A}$ can be written as

$$a = a\Pi_1 + \cdots + a\Pi_m, \qquad a\Pi_i \in \mathcal{J}_i.$$

On the other hand, if $a_i \in \mathcal{J}_i$ and

$$a_1 + \cdots + a_m = 0, \tag{3.9}$$

then $a_i = b_i\Pi_i$ for some $b_i \in \mathcal{A}$. Multiplying (3.9) by Π_i from the right and using (3.8) shows that $a_i = b_i\Pi_i = 0$ for all $i = 1, \cdots, m$. ☐

It follows from Theorems 3.1 and 3.4 that any decomposition of unity (2.4) can be obtained from a decomposition of the group algebra into minimal left ideals (as in Theorem 3.2), namely, take the sum of some of those ideals to form bigger ideals. The decomposition of $\mathcal{A}$ into minimal left ideals is not unique. We only know that in two decompositions, the ideals have to be equivalent in the sense of (3.5). This is an advantage since it allows more flexibility. It is also a disadvantage since it is not clear how to choose the minimal left ideals. This nonuniqueness is removed by considering two-sided ideals instead.

THEOREM 3.5. [12, p. 103] *Let the group algebra $\mathcal{A}$ be the direct sum of minimal left ideals $\mathcal{J}_i$, $i = 1, \cdots, m$. For each equivalence class of left ideals $\mathcal{J}_i$, let $\mathcal{I}_j$ be the direct sum of all left ideals in that class. Then*

$$\mathcal{A} = \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_\ell,$$

and each $\mathcal{I}_j$ is a minimal two-sided ideal.

THEOREM 3.6. [12, p. 104] *The decomposition of $\mathcal{A}$ into a direct sum of minimal two-sided ideals is unique up to renumbering the components.*

Theorems 3.7 and 3.8 characterize the structure of the algebra of a finite group.

THEOREM 3.7. [12, p. 107] *In any decomposition of $\mathcal{A}$ into minimal left ideals, the number of minimal left ideals equivalent to one minimal left ideal $\mathcal{J}_i$ equals the dimension of $\mathcal{J}_i$. In particular, the dimension of any two-sided ideal $\mathcal{I}$ is k^2, where k is the number of minimal left ideals in any decomposition of $\mathcal{I}$ into minimal left ideals.*

THEOREM 3.8. [12, p. 110] *The restriction of $\mathcal{A}$ to a minimal two-sided ideal $\mathcal{I}$ is isomorphic with the algebra of all linear mappings on $\mathcal{I}$.*

Suppose a basis of $\mathcal{A}$ is chosen so that each minimal two-sided ideal is spanned by a subset of the basis. Then Theorems 3.5 and 3.8 imply that in this basis, the space of left multiplications by the elements of $\mathcal{A}$ is isomorphic to the space of all block diagonal matrices where each diagonal bock corresponds to one two-sided ideal. Such a basis is called a *symmetry basis* of $\mathcal{A}$. An algorithm for the construction of a symmetry basis is not known in the general (non-commutative) case, but methods are known for certain special groups [12, p. 112]. In particular, symmetry bases are known for all crystallographic groups, which covers perhaps all two-dimensional cases important in practice [14].

If the group $\mathcal{G}$ is commutative, then two-sided and left ideals coincide and can be called simply *ideals*. Theorems 3.3–3.6 then give the following.

THEOREM 3.9. *Let $\mathcal{G}$ be a finite commutative group. Then its group algebra $\mathcal{A}$ can be decomposed into the sum of minimal ideals in a unique way up to renumbering. If $\hat{\Pi}_i$ are the idempotents corresponding to this decomposition, then any decomposition (2.3)–(2.6) can be obtained by adding subsets of the set of projections $\hat{\Pi}_i$.*

It is known that over the complex field, all minimal ideals of the algebra of a commutative group are one-dimensional [12, p. 81]. The group algebra $\mathcal{A}$ in such a case is isomorphic to the space of all diagonal matrices of order $|\mathcal{G}|$. Also, the number of minimal left ideals in the decomposition equals $|\mathcal{G}|$.

A construction of the symmetry basis and associated projections for a commutative group can be found in §6.

4. Fundamental domains. This section studies properties of a group $\mathcal{G}$ of mappings of a linear space V induced by the action of a group of mappings of the underlying domain Ω. Let , be a group of bijections of the domain Ω, and V a space of functions on Ω such that

$$f \circ \gamma \in V \qquad \forall f \in V, \gamma \in , . \tag{4.1}$$

Remark 4.1. *Note that the definition of the space V may involve boundary conditions. The condition (4.1) thus may limit the possible choices of the group , to a subgroup of the group of all bijections of Ω.*

For $\gamma \in ,$, we then may define the *action* of γ by

$$G_\gamma f = f \circ \gamma, \tag{4.2}$$

and the group

$$\mathcal{G} = \{G_\gamma : \gamma \in , \}.$$

The mapping

$$\gamma \mapsto G_\gamma$$

is isomorphism of , and $\mathcal{G}$. Define the equivalence of two points $x, y \in \Omega$ by

$$x \equiv y \iff \exists \gamma \in , : \gamma x = y.$$

The domain Ω then splits into classes of equivalence (called transitivity classes in group theory). Choosing one representative from each, a set $\Phi \subset \Omega$ is obtained from which (almost) all the domain Ω can be reconstructed by the mappings $G \in ,$ according to

$$\Omega = \bigcup_{G \in \mathcal{G}} G\Phi.$$

However, we wish to refrain from studying the existence of such a set Φ with reasonable topological properties and its theoretical construction. In most cases of practical interest, we know a-priori that the domain Ω splits into a number of subdomains, which are mapped onto each other by the mappings in , , and boundaries between those subdomains.

Assume the existence of *fundamental domains* satisfying the following definition.

Definition 4.1. *Let $\Omega_1, \ldots, \Omega_n \subset \Omega$ be such that*

(i) *For all Ω_i, no two $x, y \in \Omega_i$ can be mapped into each other by some $\gamma \in \mathcal{G}$, $\gamma \neq I$.*
(ii) *For all $\gamma \in \mathcal{G}$, and for all Ω_i, there exists a $\Omega_j = \gamma(\Omega_i)$.*
(iii) *For all pairs Ω_i and Ω_j, there exists a $\gamma \in \mathcal{G}$ such that $\Omega_j = \gamma(\Omega_i)$.*

The domains Ω_i satisfying (i)–(iii) *are called fundamental domains. Denote*

$$\tilde{\Omega} = \bigcup_{i=1}^{n} \Omega_i.$$

In Definition 4.1, each part has a certain meaning.

(i) The Ω_i are not "too large".
(ii) The Ω_i's are images of one another under the mappings from the group $\mathcal{G}$.
(iii) The Ω_i are not "too small". Specifically, (iii) prohibits breaking fundamental domains into smaller fundamental domains.

The theorems below require only the assumptions (i)–(iii). For the interpretation, however, it is important that the subdomains Ω_i cover almost all of Ω. In the continuous case, it is usual to assume that all Ω_i are open and that

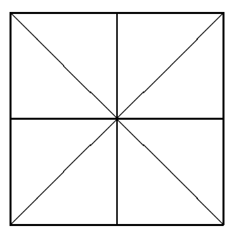
$$\Omega = \overline{\tilde{\Omega}}.$$

In the discrete case, Ω is a discrete mesh and it is natural to assume that the domains Ω_i are such that they are maximal with the properties (i)–(iii) above; that is, if $\hat{\Omega}_i$ satisfy (i)–(iii) and $\Omega_i \subset \hat{\Omega}_i$ for all i, then $\Omega_i = \hat{\Omega}_i$.

In the continuous case, the values of functions on the boundary of Ω_i is then understood as continuous extension from Ω_i or in the sense of traces. In the discrete case, the "boundary" values follow from the definition of the subspace as the range of a projection.

The question arises, when can the solution of the problem (2.1) be reduced to subproblems associated with one or few fundamental domains.

EXAMPLE 4.1. Consider the group , $= \mathbf{D}_4$ of symmetries of a square. The fundamental domains are eight triangles with one common vertex at the center of the square:



EXAMPLE 4.2. The system of fundamental domains need not be unique. Let $\mathcal{G}$ be the rotations of a circle by $\frac{2\pi k}{n}$, $k = 0, 1, \cdots, n-1$. Then the fundamental domains can be chosen as any system of nonoverlapping segments of arc size $\frac{2\pi}{n}$.

From now on, let , be a fixed group of automorphisms of Ω and $\{\Omega_i\}_{i=1}^n$ a collection of fundamental domains. The following series of statements analyze the structure of the automorphisms of Ω under the assumptions formulated in Definition 4.1. The main observation is that *every* $\gamma \in$, *can be interpreted as a permutation of fundamental domains.*

LEMMA 4.1. *If* $\gamma_1, \gamma_2 \in$, *and* $\gamma_1(\Omega_i) = \gamma_2(\Omega_i)$, *then* $\gamma_1|_{\Omega_i} = \gamma_2|_{\Omega_i}$.

Proof. If $x \in \Omega_i$ and $y_1 = \gamma_1(x)$, $y_2 = \gamma_2(x)$, from Definition 4.1(ii) there is a Ω_j such that $y_1 \in \Omega_j$ and $y_2 \in \Omega_j$. But, $y_1 = \gamma_1\gamma_2^{-1}y_2$, and $\gamma_1\gamma_2^{-1} \in \mathcal{G}$, contradicting Definition 4.1(i). □

DEFINITION 4.2. *Let* $\gamma \in$, *and* $\gamma(\Omega_i) = \Omega_j$. *Denote*

$$M_{ij} = \gamma|_{\Omega_i} : \Omega_i \to \Omega_j.$$

Lemma 4.1 shows that this definition is correct; namely M_{ij} does not depend on the choice of γ, should there be more such $\gamma \in ,$ such that $\gamma(\Omega_i) = \Omega_j$.

Now every $\gamma \in ,$ will be shown to be equal to a composition of a permutation and the mappings M_{ij}.

DEFINITION 4.3. *For $\gamma \in ,$, let π_γ be the permutation*

$$\pi_\gamma : i \mapsto j \quad \text{if } \gamma(\Omega_i) = \Omega_j.$$

Lemma 4.1 gives

LEMMA 4.2. *The mapping $\gamma \mapsto \pi_\gamma$ is a faithful representation of the group , in the group P_n of all permutations of order n, where n is the number of fundamental domains.*

From Definition 4.3 and Lemma 4.1, we have

THEOREM 4.1. *There exist one-to-one and onto mappings M_{ij} such that if $\gamma \in , : \Omega_i \to \Omega_j$, then*

$$\gamma|_{\Omega_i} = M_{ij}, \quad j = \pi_\gamma(i),$$

The assumptions in Definition 4.1 determine the number of fundamental domains.

THEOREM 4.2. *The order of the group , equals the number of fundamental domains.*

Proof. Let n be the number of fundamental domains. From Definition 4.1(ii), for any $\gamma_1 \in ,$ there is an $\Omega_j = \gamma_1(\Omega_1)$. If $\gamma_2 \in ,$ and $\Omega_j = \gamma_2(\Omega_1)$, then by Theorem 4.1, $\gamma_1|_{\Omega_1} = \gamma_2|_{\Omega_1}$ and $\gamma_1\gamma_2^{-1} = I|_{\Omega_1}$. But, by Definition 4.1(i), this means that $\gamma_1\gamma_2^{-1} = I$, so $\gamma_1 = \gamma_2$. Thus, $|,| \leq n$. Conversely, Definition 4.1(iii) gives $|,| \geq n$. □

If f is a function on Ω, define

$$f_i = f|_{\Omega_i} \circ M_{1i}.$$

Then f can be associated with an n–tuple of functions f_i on Ω_1,

$$f \approx (f_i)_{i=1}^n, \quad f|_{\Omega_i} = f_i \circ M_{i1}. \tag{4.3}$$

REMARK 4.2. *The decomposition in (4.3) ignores the values on the boundaries of the subdomains Ω_i. For our purposes of obtaining restrictions on the possible structure of the group $\mathcal{G}$, this is sufficient. For more details about the boundary conditions, see Section 9 below.*

The next theorem shows that the action $f \mapsto G_\gamma f$ of $\gamma \in ,$ on f is then simply a permutation of the components of f_i.

THEOREM 4.3. *Let $g = G_\gamma f$. Then*

$$g_i = f_j, \quad j = \pi_\gamma(i), \ i = 1, \cdots, n.$$

Proof: From Theorem 4.1 and the definition of G_γ,

$$g|_{\Omega_i} = (f \circ \gamma)|_{\Omega_i} = f|_{\Omega_j} \circ \gamma|_{\Omega_i} = f|_{\Omega_j} \circ M_{ij},$$

with $j = \pi_\gamma(i)$, so

$$g_i = g|_{\Omega_i} \circ M_{1i} = f|_{\Omega_j} \circ M_{ij} \circ M_{1i} = f|_{\Omega_j} \circ M_{1j}. \qquad \square$$

5. Prolongation from fundamental domain. Because we want to use the isomorphism of the algebra of linear operators spanned by $\mathcal{G}$ and the group algebra $\mathcal{A}$, we need to assume that the operators in $\mathcal{G}$ are linearly independent, cf., (3.3). The following simple observation states that if Ω contains "sufficiently many" points, then $\mathcal{G}$ is linearly independent.

LEMMA 5.1. *Assume that there exist points $x_j \in \Omega_1$, $j = 1, \cdots, n$, such that for any $i = 1, \cdots, n$, there exists $u_i \in V$ such that $u_i(x_j) = 0$ if $i \neq j$, $u_i(x_i) \neq 0$. Then $\mathcal{G}$ is a linearly independent set of linear operators on V.*

As before, it will be assumed that $\mathcal{G}$ is linearly independent.

Let $\Pi = \sum_{\gamma} c(\gamma) G_\gamma$ be a projection in V and $W = \operatorname{Range} \Pi \subseteq V$. Let $L : V \to V$, $LW \subseteq W$, and $f \in W$. To solve the problem

$$Lu = f, \quad u \in W$$

efficiently, u should be determined by its values on one or more fundamental domains Ω_i. Let $n = |, |$, equal to the number of fundamental domains (c.f., Theorem 4.2). Let $u \approx (u_i)_{i=1}^n$, as in (4.3), and $u \in \operatorname{Range} \Pi$, that is,

$$u = \Pi u. \tag{5.1}$$

Then, from Theorem 4.3 and (5.1),

$$u_i = \sum_{\gamma} c(\gamma) u_{\pi_\gamma(i)} = \sum_{\gamma} c(\gamma) \sum_{j} P_{ij}^{\gamma} u_j, \qquad i = 1, \ldots, n, \tag{5.2}$$

where the numbers P_{ij}^{γ} defined by

$$P_{ij}^{\gamma} = \begin{cases} 1 & \text{if} \quad j = \pi_\gamma(i) \\ & \\ 0 & \text{if} \quad j \neq \pi_\gamma(i) \end{cases}$$

form a permutation matrix $P_\gamma = (P_{ij}^{\gamma})$.

LEMMA 5.2. *The set of matrices P_γ, $\gamma \in$, , with the operation of matrix multiplication, forms a linearly independent group $\mathcal{P}$ which is isomorphic to the group , . Consequently, the algebra spanned by $\mathcal{P}$ is isomorphic to the group algebra $\mathcal{A}$.*

Proof. Lemma 4.2 and known facts about permutation matrices give that , is isomorphic to $\mathcal{P}$. That $\mathcal{P}$ is linearly independent follows from Definition 4.1 (ii) and Theorem 4.2, which imply that the first columns of the $n \times n$ matrices P_γ are the n unit coordinate vectors. □

The next theorem is concerned with using the isomorphism of , and $\mathcal{P}$ to generate the range of a projection Π of the form (2.6).

THEOREM 5.1. *If the dimension of the range of a projection*

$$\Pi = \sum c(\gamma) G_\gamma$$

in the group algebra $\mathcal{A}$ is d, then there exist indices $\{i_1, \cdots, i_d\} \subseteq \{1, \cdots, n\}$ such that the values of every $u \in \operatorname{Range} \Pi \subseteq V$ on $\tilde{\Omega}$ are uniquely determined by the values of the functions $u_{i_1}, \cdots, u_{i_d}$.

Proof. As a consequence of Lemma 5.2, the rank of the matrix $I - \sum c(\gamma) P_\gamma$ is $n - d$. Thus, there exist $n - d$ linearly independent rows of $I - \sum c(\gamma) P_\gamma$. Choosing $i_1, \cdots, i_d$ to be indices of the remaining d rows, the system of linear equations for real unknowns v_j,

$$\sum_{j=1}^{n} \Big(I - \sum c(\gamma) P_\gamma \Big)_{ij} v_j = 0, \quad i \notin \{i_1, \cdots, i_d\}, \tag{5.3}$$

has a unique solution for v_i as a linear combination of $v_{i_1}, \cdots, v_{i_d}$. For $x \in \Omega_1$, where Ω_1 is the fundamental domain, we apply this to $v_j = u_j(x)$. □

REMARK 5.1. *The theorem and equation (5.3) uniquely determine the prolongation P_i.*

We are now ready for the main result of this section.

THEOREM 5.2. *Let , be a group of one-to-one, onto mappings of Ω with fundamental domains $\Omega_1, \cdots, \Omega_n$, and $\mathcal{G}$ the induced group of automorphisms G_γ of a linear space V of functions on Ω. Let Π_i, $i = 1, \cdots, m$, be projections*

$$\Pi_i = \sum_{\gamma \in \Gamma} c_i(\gamma) G_\gamma$$

such that

$$I = \sum_{i=1}^{m} \Pi_i, \qquad \Pi_i \Pi_j = 0, \quad i \neq j.$$

Assume that $\mathcal{G}$ is linearly independent and that the values on $\tilde{\Omega}$ of every function $u \in V_i \equiv \operatorname{Range} \Pi_i$ are uniquely determined by the restriction of u to some fundamental domain $\Omega_{k(i)}$. Then the group , is commutative and $m = n = |, |$.

REMARK 5.2. *The assumption that $\mathcal{G}$ is linearly independent is only technical and it will be satisfied if the space V contains sufficiently many functions. It will always be satisfied in practice, cf., Lemma 5.1.*

Proof. It follows from Theorem 5.1 that the dimension of the range of every Π_i in the algebra $\mathcal{A}$ is one. The group , is isomorphic to the group of right multiplications on $\mathcal{A}$, which consists of linear mappings γ_R on $\mathcal{A}$, defined by

$$\gamma_R : \sum_{\alpha \in \Gamma} c(\alpha) G_\alpha \mapsto \left(\sum_{\alpha \in \Gamma} c(\alpha) G_\alpha \right) G_\gamma.$$

It holds that

$$\mathcal{A} = \hat{V}_1 \oplus \cdots \oplus \hat{V}_n,$$

where $\hat{V}_i$ is the range of Π_i in $\mathcal{A}$. Since for any $a \in \hat{V}_i$, $a = \Pi_i a$ and thus, $a\gamma = \Pi_i a \gamma$, it holds that $\gamma_R \hat{V}_i \subset \hat{V}_i$ for any $\gamma \in$, and all $i = 1, \cdots, n$.

Choosing a nonzero vector in each $\hat{V}_i$ gives a basis of $\mathcal{A}$ such that all transformations from $\mathcal{G}$ are represented by diagonal matrices in that basis. It remains to note that multiplication of diagonal matrices is commutative. □

6. Commutative Group Case. The same result as in Theorem 5.2 can be derived without using mappings of the domain once the number of subspaces is known to equal $|\mathcal{G}|$.

THEOREM 6.1. *Let $\mathcal{G}$ be a linearly independent finite group of linear transformations on V, with $V = V_1 \oplus \cdots \oplus V_n$, where $n = |\mathcal{G}|$, each subspace V_i is the range of a projection Π_i, and the projections Π_i are linear combinations of the transformations in $\mathcal{G}$. Then $\mathcal{G}$ is commutative.*

Proof. In the group algebra $\mathcal{A}$, $\mathcal{J}_i = \mathcal{A}\Pi_i$ is a left ideal, and $\mathcal{A} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_n$. Since the dimension of $\mathcal{A}$ is n, the dimension of each left ideal $\mathcal{J}_i$ is one. Choosing a nonzero vector in each $\mathcal{J}_i$ gives a basis of $\mathcal{A}$ in which all transformations from $\mathcal{G}$ are represented by diagonal matrices. □

Thus, for a large class of domain reduction algorithms, the underlying group must be commutative. The next theorem shows that the decomposition is then essentially unique.

THEOREM 6.2. *Let the group $\mathcal{G}$ be commutative. Then there is a unique decomposition of V into the direct sum $V = V_1 \oplus \cdots \oplus V_n$ (up to renumbering of the components) such that the associated projections Π_i are linear combinations of the mappings G_γ, $\gamma \in \mathcal{G}$, and the number n of components is the maximal possible, $n = |\mathcal{G}|$.*

Proof. Since n is the largest possible, the ranges of the projections Π_i in the group algebra $\mathcal{A}$ are minimal left ideals. Because the group $\mathcal{G}$ is commutative, they are two-sided ideals, and, according to Theorem 5.1, the dimensions of the ranges of the projections in the group algebra are one. Hence, from Theorem 3.9, the decompositions into subspaces and the projections Π_i are unique up to renumbering. □

Now, we show explicitly how to build the projections for the commutative case. Let $\{B_1, \cdots, B_q\} \subset \mathcal{G}$ be generators of the group , , i.e., every element $G \in \mathcal{G}$ can be expressed uniquely as

$$G = B_1^{k_1} \cdots B_q^{k_q}, \qquad 0 \le k_j \le d_j - 1,\ 1 \le j \le n, \tag{6.1}$$

where d_j is the order of the generator B_j,

$$B_j^{d_j} = I, \qquad B_j^l \ne I, \qquad \forall l = 1, \cdots, d_j - 1.$$

For $j = 1, \cdots, q$, define the polynomials p_{jk}, $k = 0, \cdots, d_j - 1$, of degree $d_j - 1$ by

$$p_{jk}(e^{\frac{2\pi i l}{d_j}}) = \delta_{kl}, \qquad k, l = 0, \cdots, d_j - 1. \tag{6.2}$$

Then for any $j = 1, \cdots, q$,

$$\sum_{k=0}^{d_j - 1} p_{jk}(\lambda) = 1. \tag{6.3}$$

It is easy to see that the coefficients of the polynomials p_{jk} are coefficients of the discrete Fourier transform. Write

$$p_{jk}(\lambda) = \sum_{m=0}^{d_j - 1} p_{jkm} \lambda^m.$$

Then from (6.2),

$$\sum_{m=0}^{d_j-1} p_{jkm} e^{\frac{2\pi i l m}{d_j}} = \delta_{kl} \qquad k, l, m = 0, \cdots d_j - 1,$$

which gives

$$p_{jkm} = \frac{1}{d_j} \sum_{l=0}^{d_j-1} \delta_{kl} e^{-\frac{2\pi i l m}{d_j}} = \frac{1}{d_j} e^{-\frac{2\pi i k m}{d_j}}. \tag{6.4}$$

From (6.3), the polynomials p_{jk} form a family of decompositions of unity, and we can use them to define a decomposition of the space V into a direct sum and the corresponding decomposition of identity into projections. Because we are looking for a decomposition into $|\mathcal{G}|$ subspaces, we can index the subspaces and the projections by the elements of $\mathcal{G}$. For $G \in \mathcal{G}$, and B_j and k_j as in (6.1), define the linear mapping

$$\Pi_G = p_{1,k_1}(B_1) \cdots p_{q,k_q}(B_q) \tag{6.5}$$

and the subspace V_G of V is given by (6.1).

$$V_G = \{u \in V : \ B_j u = e^{\frac{2\pi i k_j}{d_j}} u, \forall j = 1, \cdots, q\}. \tag{6.6}$$

Note that using (6.4), (6.5) can be written as

$$\Pi_G = \prod_{j=1}^{q} \frac{1}{d_j} \sum_{m=1}^{d_j-1} B_j^m e^{-\frac{2\pi i k m}{d_j}} = \frac{1}{|\mathcal{G}|} \sum_{T \in \mathcal{G}} \chi_G(T) T, \tag{6.7}$$

where, for $G \in \mathcal{G}$ given by (6.1), $\chi_G(T)$ is the group character

$$\chi_G(T) = e^{-2\pi i \left(\frac{k_1 m_1}{d_1} + \cdots + \frac{k_q m_q}{d_q}\right)}, \qquad T = B_1^{m_1} \cdots B_q^{m_q}.$$

THEOREM 6.3. *For any $G \in \mathcal{G}$, the mapping Π_G is a projection onto the subspace V_G. Further it holds that*

$$\Pi_G \Pi_H = 0, \qquad \forall G, H \in \mathcal{G}, G \neq H, \tag{6.8}$$

$$\sum_{G \in \mathcal{G}} \Pi_G = I, \tag{6.9}$$

and

$$V = \bigoplus_{G \in \mathcal{G}} V_G. \tag{6.10}$$

Proof. From $B_j^{d_j} = I$, it follows that all of the eigenvalues of B_j are just $e^{\frac{2\pi i l}{d_j}}$, $l = 0, \cdots, d_j - 1$. It follows from (6.2) that $p_{jk}^2 = p_{jk}$ on the spectrum of B_j, and thus $p_{jk}(G)p_{jk}(G) = p_{jk}(G)$. Using commutativity of $\mathcal{G}$, $\Pi_G\Pi_G = \Pi_G$, so that Π_G is a projection. That V_G is the range of Π_G follows easily again from commutativity of $\mathcal{G}$. If $G \neq H$, then $\Pi_G = p_{1,k_1}(B_1)\cdots p_{q,k_q}(B_q)$ and $\Pi_H = p_{1,l_1}(B_1)\cdots p_{q,l_q}(B_q)$, and there exist at least one j such that $k_j \neq l_j$. Then $p_{j,k_j}p_{j,l_j} = 0$ on the spectrum of B_j, and (6.8) follows. Equation (6.9) follows immediately from (6.3) and commutativity of $\mathcal{G}$. Finally, (6.10) is an immediate consequence of (6.8), (6.9), and the fact that the Π_G's are projections. □

Since the number of components in the decomposition (6.9) is the maximal possible, we have constructed the unique decomposition from Theorem 6.2.

The automorphisms $G \in \mathcal{G}$ were used to construct the projections Π_i. The following lemma shows how to reconstruct the automorphisms from the projections.

LEMMA 6.1. *Let $\mathcal{G}$ be commutative. Then* $B_j = \sum_{G\in\mathcal{G}} e^{\frac{2\pi i k_j}{d_j}}\Pi_G$.

Proof. This is the spectral decomposition of B_j as a linear operator on the group algebra $\mathcal{A}$. It suffices to note that from the definition of of V_G in (6.6),

$$B_j|_{V_G} = e^{\frac{2\pi i k_j}{d_j}} I|_{V_G}$$

and use (6.9). □

The following theorem follows immediately.

THEOREM 6.4. *If the group $\mathcal{G}$ is commutative and $L : V \to V$ is a linear mapping, then L commutes with all Π_G if and only if it commutes with all $G \in \mathcal{G}$.*

7. Nested Decompositions. Now consider an application of the same decomposition method recursively. Some or all of the spaces V_G are further decomposed using automorphisms of those spaces, which results in a similar decomposition as before, but with a new commutative group $\mathcal{G}$.

THEOREM 7.1. *For any $G \in \mathcal{G}$, let H_G be an automorphism of V_G. Then,*

$$\tilde{H} = \sum_{G\in\mathcal{G}} \Pi_G H_G \Pi_G \tag{7.1}$$

is an automorphism of V, which commutes with all $G \in \mathcal{G}$. Moreover, if $\mathcal{H}_G$ is a commutative group of automorphisms for each G, then

$$\mathcal{H} = \{\tilde{H} : H_G \in \mathcal{H}_G \ \forall G \in \mathcal{G}\}$$

is a commutative group of automorphisms, and $\mathcal{H} \cup \mathcal{G}$ generates a commutative group of automorphisms of V.

Proof. To show that the mapping $\tilde{H}$ is one-to-one, note that

$$\tilde{H}v = 0, \quad v = \sum_{G\in\Gamma} v_G, \quad G \in V_G \qquad \Rightarrow \qquad H_G v_G = 0 \quad \forall G \in \mathcal{G},$$

thus $v_G = 0$ because the mappings H_G are one-to-one. To show that $\tilde{H}$ is onto, note that

$$v \in V, \quad v = \sum_{G\in\mathcal{G}} v_G, \quad v_G \in V_G \qquad \Rightarrow \qquad v = \tilde{H} \sum_{G\in\mathcal{G}} H_G^{-1} v_G.$$

To show that $\tilde{H}$ commutes with $\mathcal{G}$, we compute

$$\tilde{H}G_j = \sum_{G\in\mathcal{G}} H_G\Pi_G G_j = \sum_{G\in\mathcal{G}} G_j|_{V_G} H_G \Pi_G = G_j\tilde{H},$$

since G_j reduces to scalar multiplication on V_G. The rest of the proof is obvious. □

Theorem 7.1 suggests that any decomposition of V into a direct sum should be induced by a commutative automorphism group. Theorem 7.2 shows that this is indeed the case.

THEOREM 7.2. *If $V = \bigoplus_{i=1}^m W_i$, where the W_i are closed, then there exists a commutative group $\mathcal{G}$ of automorphisms of V such that each subspace W_i is one of subspaces V_G, $G\in\mathcal{G}$, defined in (6.6).*

Proof. Let $I = \sum_{i=1}^m \Pi_i$ be the decomposition of unity associated with $V = \bigoplus_{i=1}^m W_i$, where Π_i is the projection of W_i. Since the subspaces W_i are closed, the projections Π_i are continuous. Define

$$B_i = \Pi_i - \prod_{\substack{j=1\\ j\neq i}}^m \Pi_j.$$

Then $B_i^2 = I$. Let $\mathcal{G}$ be the group generated by B_i, $i = 1,\cdots,m$. If $G = B_1^{\sigma_1}\cdots B_m^{\sigma_m}$, $\sigma_j \in \{0,1\}$, then

$$V_G = W_j \quad\Longleftrightarrow\quad G_j u = \sigma_j u \quad\Longleftrightarrow\quad \begin{cases} u\in V_j, & \forall\sigma_j = 1,\\ \\ u\notin V_j, & \forall \sigma_j = 0.\end{cases}$$

Thus, $V_G = W_j$ if $\sigma_j = 1$, $\sigma_k = 0$, $k\neq j$. Otherwise, $V_G = \{0\}$. □

REMARK 7.1. *It follows from Theorems 6.4 and 7.2 that in decomposing a space V, the projections commute with L if and only if there exists a commutative group of automorphisms of V which generates the decomposition and commutes with L.*

§4 above was concerned with the properties of the group $\mathcal{G}$ of automorphisms of V induced by a group , of bijections of the domain Ω. The group , was not necessarily commutative. The group constructed by the nested decomposition of this section is always commutative, and so it follows that it cannot always be obtained as a group of actions of bijections of Ω.

The question which automorphisms of V can be induced by the action of a bijection of the domain thus arises naturally.

For an automorphism $G: V\to V$ and $x\in\Omega$, denote

$$\hat{G}(x) = \bigcap_{u\in V}\{y\in\Omega : (Gu)(x) = u(y)\}. \tag{7.2}$$

THEOREM 7.3. *Let V be a linear space of real or complex functions on Ω. Then the following holds:*

(i) If γ is a bijection of Ω such that $G = G_\gamma$, and V divides the points of Ω, that is,

$$\forall x_1, x_2 \in \Omega, x_1 \neq x_2 \; \exists u \in V: \quad u(x_1) \neq u(x_2). \tag{7.3}$$

then

$$\hat{G}(x) = \{\gamma(x)\}, \qquad \forall x \in \Omega. \tag{7.4}$$

In particular, G can be an action of a bijection of Ω only if $|\hat{G}(x)| = 1$ for all $x \in \Omega$.

(ii) If $\mathcal{G}$ is a group of automorphisms of V and

$$|\hat{G}(x)| = 1, \qquad \forall x \in \Omega, \quad \forall G \in \mathcal{G}, \tag{7.5}$$

then for any $G \in \mathcal{G}$, the mapping $\gamma = \gamma(G) : \Omega \to \Omega$ defined by (7.4) is a bijection of Ω and $G = G_{\gamma(G)}$.

Proof. (i) If $\gamma(x) = y$, then $(G_\gamma u)(x) = u(\gamma(x))$ for all $u \in V$, thus $\gamma(x) \in \hat{G}_\gamma(x)$. If

$$y_1, y_2 \in \hat{G}_\gamma, \qquad y_1 \neq y_2,$$

then

$$(G_\gamma u)(x) = u(y_i), \quad i = 1, 2, \qquad \forall u \in V.$$

From (7.3) there is a $u \in V$ such that $u(y_1) \neq u(y_2)$, which is a contradiction.

(ii) Let $G \in \mathcal{G}$. From (7.5), (7.4) defines a mapping γ of Ω. To show that γ is one-to-one and onto, consider the mapping $H = G^{-1} \in \mathcal{G}$. From (7.2), it follows that

$$\begin{aligned} y \in \hat{G}(x) \qquad &\Longleftrightarrow \qquad \forall u \in V : (Gu)(x) = u(y), \\ x \in \hat{H}(y) \qquad &\Longleftrightarrow \qquad \forall v \in V : (Hv)(y) = v(x). \end{aligned}$$

Thus $y \in \hat{G}(x)$ if and only if $x \in \hat{H}(y)$. Consequently, the mapping χ defined by $\hat{H}(y) = \{\chi(y)\}$ is the inverse of γ. $\square$

Using this theorem, the following example shows that the nested decomposition in general gives raise to automorphisms which are not induced by bijections of the domain.

EXAMPLE 7.1. Let

$$\Omega = (-4, 4), \qquad V = \text{space of all bounded functions on } \Omega,$$

and

$$, = \{I, G\}, \qquad G : u(x) \mapsto u(-x).$$

Following the construction in Section 6, we obtain the projections

$$\Pi_I = \frac{1}{2}(I + G) \qquad \text{and} \qquad \Pi_G = \frac{1}{2}(I - G)$$

and the subspaces

$$V_I = \{u : u(x) = u(-x)\} \qquad \text{and} \qquad V_G = \{u : u(x) = -u(-x)\}.$$

Obviously, G is defined by a bijection of the domain,

$$\gamma : x \mapsto -x.$$

Let H be defined from the mapping

$$\chi(x) : x \mapsto \begin{cases} -x - 2 & x < 0. \\ 0 & x = 0, \\ 2 - x & x > 0, \end{cases}$$

Define

$$H : V_G \mapsto V_G \text{ by } Hu = u \circ \chi.$$

Then $H^2 = I$. From Theorem 7.1, the mapping

$$K = \Pi_G H \Pi_G + \Pi_I$$

commutes with G. However, K cannot be induced by any mapping of $\Omega \mapsto \Omega$. Let $u \in V$ be defined by $u(-3) = 1$ and $u = 0$ elsewhere. Note that all of the functions involved in evaluating $\tilde{H}u$ (see (7.1)) are zero everywhere except $\Omega \setminus \{-3, -1, 1, 3\}$:

Functions	x			
	-3	-1	1	3
u	1	0	0	0
$\Pi_I u$	.5	0	0	.5
$\Pi_G u$	.5	0	0	$-.5$
$H\Pi_G u$	0	.5	$-.5$	0
Ku	.5	.5	$-.5$	.5

Thus, $Ku \neq 1$ in Ω, so $\hat{K}(1) = \emptyset$ (see (7.2)) . Hence, according to Theorem 7.3, K cannot be the action of a bijection of Ω. A similar example can be constructed quite easily for the case when V is the space of all continuous functions on Ω.

8. Comparison with the theory of [1] and [2] . Allgower, Böhmer, and Mei [1] assume that the operator L commutes with all g in a group $\mathcal{B}$ of automorphisms of V, but L is not necessarily linear. The transformations $g \in \mathcal{B}$ are of the form

$$g : u \mapsto \sigma u \circ \gamma, \tag{8.1}$$

where $\sigma = \pm 1$ and γ is an isometric bijection of the domain Ω. Thus,

$$\mathcal{B} = \mathcal{G} \times \mathbf{Z}_2,$$

where $\mathcal{G}$ is a group of actions of bijections of Ω as in our approach and $\mathbf{Z}_2 = \{1, -1\}$.

For a subgroup $\mathcal{S} \subset \mathcal{B}$, its *fixed point subspace* is defined by

$$V^{\mathcal{S}} = \{u \in V : \forall g \in \mathcal{S} : gu = u\} \tag{8.2}$$

and the projection $\Pi_{\mathcal{S}}$ on $V^{\mathcal{S}}$ is defined by

$$\Pi_{\mathcal{S}} = \frac{1}{|\mathcal{S}|} \sum_{g \in \mathcal{S}} g. \tag{8.3}$$

Now if the right hand side $f \in V^{\mathcal{S}}$ for a subgroup $\mathcal{S}$, then the solution of $Lu = f$ is also $u \in V^{\mathcal{S}}$.

If $\mathcal{S}_i$, $i = 1, \cdots, m$, is a family of subgroups of $\mathcal{B}$ such that for the associated subspaces V_i,

$$V = V_1 \oplus \cdots \oplus V_m, \qquad V_i = V^{\mathcal{S}_i}, \tag{8.4}$$

and if L is linear, then one can proceed with the decomposition of the problem $Lu = f$ as in §2 above.

It is easy to see that this approach can be considered to be a special case of the theory developed in this paper, because the projections $\Pi_{\mathcal{S}_i}$ in (8.4) are linear combinations of mappings in $\mathcal{G}$ (with coefficients $\pm 1/|\mathcal{S}|$) and thus generate a decomposition of the group algebra $\mathcal{A}$ into left ideals $\mathcal{A}\Pi_{\mathcal{S}_i}$ by Theorem 3.4 (assuming the technical condition that the group $\mathcal{G}$ is linearly independent). It is also easy to see that if the subgroup $\mathcal{S}_i$ is normal, then $\mathcal{A}\Pi_{\mathcal{S}_i}$ is a two-sided ideal. Note that, in contrast with our theory, their group $\mathcal{B}$ is not linearly independent.

In [1], the group $\mathbf{D}_4$ of the symmetries of the square is considered and a decomposition into 6 subproblems is given based on 6 selected subgroups of the group $\mathbf{D}_4 \times \mathbf{Z}_2$. But it is not clear how to select suitable subgroups in the general case.

Allgower, Böhmer, Georg, and Miranda [2] generalized this approach to projections not associated with subgroups of $\mathcal{B}$ but rather with characters χ of $\mathcal{G}$. A character χ is defined in [2] as a group homomorphism of a subgroup $S_\chi \subset \mathcal{G}$ into complex units for $T \in S_\chi$, $\chi(T) = 0$ for $T \notin S_\chi$. With χ is associated the projection

$$\Pi_\chi = \frac{1}{|\mathcal{S}_\chi|} \sum_{T \in S_\chi} \chi(T) T, \tag{8.5}$$

onto the generalization of the fixed-point subspace (8.2), given by

$$V_\chi = \{u \in V : Tu = \chi(T^{-1})u, \forall T \in \mathcal{S}\}.$$

The main principle employed in [2] is that if $\mathcal{X}$ is a set of characters such that if the character set $\mathcal{X}$ forms a decomposition of unity

$$\sum_{\chi \in \mathcal{X}} \gamma_\chi \chi = \tau, \tag{8.6}$$

for some coefficients γ_χ, $\chi \in \mathcal{X}$, where τ is the trivial character, given by $\tau(I) = 1$, $\tau(T) = 0$, $T \neq I$, then

$$\sum_{\chi \in \mathcal{X}} |\mathcal{S}_\chi| \gamma_\chi \Pi_\chi = I.$$

The solution of a linear problem on the space V can then be decomposed into subproblems on the spaces V_χ, which do not necessarily form a direct sum of V. As pointed out in [2], the spaces V_χ do form a direct sum in many important cases, and the theory of this paper then applies. Suitable decompositions of unity were given in [2] for several important groups.

Note that according to (6.7), our construction of the projections in the commutative case is a special case of the construction (8.5) with summation over all characters for the whole group $\mathcal{G}$, that is, $S_\chi = \mathcal{G}$. (Such characters of a commutative group $\mathcal{G}$ form the so-called dual group.)

9. Boundary conditions. This section contains several remarks on the role of boundary conditions in the domain reduction method.

The choice of $\mathcal{G}$ is limited by the boundary conditions incorporated in V and L. For example, if V is a space of functions on a square with zero boundary values on one side s and no conditions on the other sides, the only symmetry available is the reflection about the axis perpendicular to s. A similar restriction arises if L is, say, the Laplace operator with mixed boundary conditions ($\partial u/\partial \nu + au = b$) on one side and Neumann boundary conditions on other sides.

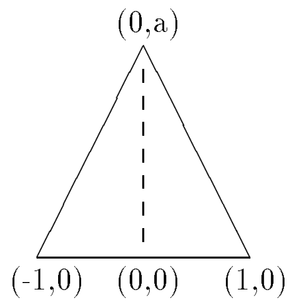
In the discrete case, the present theory applies with Ω being a mesh. Then the geometry of the mesh, in addition to its shape may further limit the group , of symmetries. Note, however, that meshes can be always constructed that do not limit any continuous symmetry [13].

The subspace V_i, given by (2.4), is represented by functions on one or more fundamental domains according to Theorem 5.1. A new boundary value problem can then be defined based on those domains with the boundary condition implied by the definition of the reduced space V_i and/or the reduced operator L_i (see (2.9)). This reduced problem can be further decomposed using its own symmetries (see §7). However, some of those reduced problems possess *less* symmetry because of the introduced boundary conditions, which may eventually prevent some reduced problem from being decomposed further.

For more details about boundary condition for reduced problems on a fundamental domain, see [3] and [6]. Munthe-Kaas [13] discusses in detail boundary conditions compatible with various groups of isometries.

Experience with parallel implementations, reported in [4, 6], shows that the method is highly efficient if a suitable large group of symmetries exists; unfortunately, the potential for parallelism is limited by the problems briefly listed above. However, two tricks overcome this limitation. The first is to extend the problem domain to one where similar boundary conditions are symmetric in some obvious manner (see [5]). The second is to find a reasonable problem similar to the original problem with symmetric boundary conditions. The approximate problem can then be used as a preconditioner in an iterative method for the solution of the original problem (see [8]).

10. Examples. EXAMPLE 10.1. Consider the *2-way decomposition of a symmetric 2 dimensional domain.* For example, a triangle with vertices at (-1,0), (1,0), and (0,a), $a > 0$,



Here,

$$G_1 : u(x, y) \mapsto u(x, y) \qquad \text{and} \qquad G_2 : u(x, y) \mapsto u(-x, y)$$

Hence,

$$\Pi_0 = \frac{1 - G_1}{2} \qquad \text{and} \qquad \Pi_1 = \frac{1 + G_1}{2}$$

G_j and Π_j, $j = 1, 2$, can be defined similarly if the symmetry had been across the x–axis instead of the y–axis.

EXAMPLE 10.2. Consider the *4-way decomposition of a square.* Let $\Omega = (-1, +1)^2$, and $\mathcal{G}$ be generated by the transformations

$$G_1 : u(x, y) \mapsto u(-x, y), \qquad G_2 : u(x, y) \mapsto u(x, -y).$$

Because G_1 and G_2 commute, the generic element of $\mathcal{G}$ is

$$G = G_1^{k_1} G_2^{k_2}, \qquad k_1, k_2 \in \{0, 1\}.$$

Using the results of §6, we have the polynomials of degree at most one defined by

$$p_{jk}((-1)^l) = \delta_{kl}, \qquad j = 1, 2, \quad k = 0, 1, \quad l = 0, 1,$$

from (6.2), which gives

$$p_{1\ell}(t) = p_{2\ell}(t) = \frac{1 + (-1)^\ell t}{2},$$

and from (6.5), we have denoting for $G = G_1^{k_1} G_2^{k_2}$, $\Pi_G = \Pi_{k_1 k_2}$, $k_1, k_2 = 0, 1$,

$$
\begin{aligned}
\Pi_{00} &= \frac{1+G_1}{2}\frac{1+G_2}{2} \\
\Pi_{01} &= \frac{1+G_1}{2}\frac{1-G_2}{2} \\
\Pi_{10} &= \frac{1-G_1}{2}\frac{1+G_2}{2} \\
\Pi_{11} &= \frac{1-G_1}{2}\frac{1-G_2}{2}
\end{aligned}
$$

This can be expanded to d dimensions in the obvious manner.

Some other interesting examples include the following.

- 3 and 4 way decompositions on a triangle
- 4 way decomposition using diagonal symmetries of a parallelogram
- Non-commutative decomposition on a square such as the 8 way domain reduction in [3]
- Laplace equation on a rectangle with periodic boundary conditions in the x-direction and the transformations equal to translations in the x-direction by a fraction of the rectangle side; then the decomposition given in this paper is equivalent to Fourier transform in the x-direction.

11. Conclusions. The domain reduction method is deeply rooted in mathematical analysis from the late nineteenth and early twentieth centuries. Previous results in [1],[3], [4], [6], [7], and [9] used various ad-hoc constructions. The present theory makes a connection with the classical group representation theory, shows that the previous constructions were in fact forced by the fabric of group representation, and characterizes all such possible constructions. Further, the results apply to results that have not been published yet ([5] and [8]).

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