Introduction to the conforming and nonconforming finite element methods

Dongwoo Sheen

Graduate School of Informatics
Kyoto University
Kyoto 608–8501, Japan

Email: dongwoosheen@gmail.com
http://www.nasc.snu.ac.kr

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Chapter 1

Introduction to conforming and nonconforming finite elements

1.1 The model problem

Throughout the lecture $\mathscr{F}$ denotes the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open region with smooth or polygonal boundary $\partial\Omega$. Consider the following boundary value problem:
given $f \in H^{-1}(\Omega)$ find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \mathscr{L}u := -\nabla\cdot\mathbf{A}(\mathbf{x})\nabla u + b(\mathbf{x})u &= f, \quad \Omega, & \text{(1.1a)} \\ u &= 0, \quad \partial\Omega. & \text{(1.1b)} \end{aligned}$$

- Uniform ellipticity assumption: $\mathbf{A}(\cdot)$ is a uniformly positive–definite $d \times d$ matrix–valued function defined on Ω: *i.e.,*

$$\begin{aligned} \langle \mathbf{A}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle &\geq c_0|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathscr{F}^d \quad a.e. \quad \mathbf{x} \in \Omega; & \text{(1.2a)} \\ b(\mathbf{x}) &\geq 0 \quad a.e. \quad \mathbf{x}\Omega. & \text{(1.2b)} \end{aligned}$$

1.1.1 Weak formulation

Multiply (1.1a) by an arbitrary $v \in H_0^1(\Omega)$ and use the Divergence Theorem, keeping in mind (1.1b). Then we have the following weak problem: to find $u \in H_0^1(\Omega)$ such that

$$(\mathbf{A}\nabla u, \nabla v) + (bu, v) = \langle f, v\rangle_{H^{-1},H_0^1} \quad \forall v \in H_0^1(\Omega). \tag{1.3}$$

Here, and in what follows, standard notations from functional analysis are used.

- The duality paring $\langle\cdot,\cdot\rangle_{V',V}$, where V and V' are a normed linear space V and its dual space V', respectively. The corresponding norms are denoted by $\|\cdot\|_V$ and $\|\cdot\|_{V'}$. The duality is defined as $\langle f, v\rangle_{V',V} = f(v)$ for all $v \in V$ such that $f : V \mapsto \mathscr{F}$ is a continuous linear functional on V. The dual norm is given by

$$\|f\|_{V'} = \sup_{0\neq v\in V} \frac{\langle f, v\rangle_{V',V}}{\|v\|_V},$$

  for all $f \in V'$.

- The space of square integrable functions $L^2(\Omega)$ on Ω is denoted by

$$L^2(\Omega) = \{v : \Omega \to \mathscr{F} \, : \, \int_\Omega |v(\mathbf{x})|^2 \, d\mathbf{x} < \infty\},$$

with the $L^2(\Omega)$–inner product:

$$(u, v) = \int_\Omega u(\mathbf{x})\overline{v(\mathbf{x})}\,d\mathbf{x} \quad \forall u, v \in L^2(\Omega)$$

and the $L^2(\Omega)$–norm:

$$\|v\| = \sqrt{\int_\Omega |v(\mathbf{x})|^2\,d\mathbf{x}} \quad \forall v \in L^2(\Omega).$$

- The Sobolev spaces, $H^m(\Omega) = \{v \in L^2(\Omega) : \|v\|_m < \infty\}$, are Hilbert spaces with the inner product

$$(u, v)_{m,\Omega} = \sum_{|\boldsymbol{\alpha}|\le m} \left(\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^\alpha}, \frac{\partial^{\boldsymbol{\alpha}} v}{\partial x^\alpha}\right).$$

Here, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$ is a multi–index with its length $|\boldsymbol{\alpha}| = \sum_{j=1}^d \alpha_j$, which is used to denote the partial derivatives

$$\frac{\partial^{\boldsymbol{\alpha}} v}{\partial x^{\boldsymbol{\alpha}}} = \frac{\partial_1^\alpha}{\partial x_1} \cdots \frac{\partial_d^\alpha}{\partial x_d} v.$$

The corresponding norms and seminorms are designated by

$$\|v\|_{m,\Omega} = \sqrt{(v, v)_{m,\Omega}} \quad \text{and} \quad |v|_{m,\Omega} = \sqrt{\sum_{|\boldsymbol{\alpha}|=m} \|\frac{\partial^{\boldsymbol{\alpha}} v}{\partial x^{\boldsymbol{\alpha}}}\|^2},$$

respectively.

- In particular, for $m = 1$,

$$H^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_j} \in L^2(\Omega), \quad j = 1, \cdots, N\},$$

is equipped with the inner product, norm, and seminorm:

$$(u, v)_{1,\Omega} = \int_\Omega u\bar{v}\,d\mathbf{x} + \sum_{k=1}^d \int_\Omega \frac{\partial u}{\partial x_k}\overline{\frac{\partial v}{\partial x_k}}\,d\mathbf{x},$$

$$\|u\|_{1,\Omega} = [(u, u)_{1,\Omega}]^{\frac{1}{2}}, \quad \text{and} \quad |u|_{1,\Omega} = [(\nabla u, \nabla u)]^{\frac{1}{2}},$$

respectively.

- $H_0^m(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the norm of $H^m(\Omega)$
- Often, $H^{-1}(\Omega)$ will mean the dual space of $H_0^1(\Omega)$ with the dual norm

$$\|f\|_{-1,\Omega} = \sup_{0\ne v\in H_0^1(\Omega)} \frac{\langle f, v\rangle_{H^{-1}(\Omega),H^1(\Omega)}}{\|v\|_{1,\Omega}}.$$

1.1.2 Unique Solvability

Set $V = H_0^1(\Omega)$ and define a bilinear form $a(\cdot,\cdot) : V \times V \mapsto \mathscr{F}$ and a linear functional $\ell : V \mapsto \mathscr{F}$ by

$$a(u, v) = (\mathbf{A}\nabla u, \nabla v) + (bu, v), \quad \text{and} \quad \ell(v) = \langle f, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} \quad \forall\, u, v \in V.$$

Then, (1.3) is rewritten as to find $u \in V$ such that

$$a(u, v) = \ell(v) \quad \forall v \in V. \tag{1.4}$$

We now state the following frequently used lemma:

Lemma 1.1 (The Lax-Milgram Lemma). *Assume the following*

1. *V is a Hilbert space;*
2. *$a(\cdot,\cdot)$ is continuous on V, i.e. there exists $C > 0$ such that*
$$|a(u,v)| \le C\|u\|_V \, \|v\|_V \quad \forall u, v \in V;$$
3. *$a(\cdot,\cdot)$ is V–coercive on V, i.e. there exists $\alpha > 0$ such that*
$$|a(u,u)| \ge \alpha \|u\|_V^2 \quad \forall u \in V;$$
4. *$\ell(\cdot)$ is continuous on V, i.e. there exists $M > 0$ such that*
$$|\ell(v)| \le M\|v\|_V \quad \forall v \in V.$$

Then, there exists a unique solution $u \in V$ to (1.4) *such that*
$$\|u\|_V \le \frac{M}{\alpha}\|\ell\|_{V'}.$$

Exercise 1.1. *Prove the above Lax–Milgram Lemma.*

1.1.3 Elliptic regularity

If the domain Ω and the coefficients $\mathbf{A}(\mathbf{x})$ and $b(\mathbf{x})$ are smooth enough and $f \in L^2(\Omega)$, then the above solution $u \in H_0^1(\Omega)$ to (1.4) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$.

Exercise 1.2. *Show that $(\mathbf{c}\cdot\nabla u, u) = -\frac{1}{2}((\nabla\cdot\mathbf{c})u, u)$ for all $u \in H_0^1(\Omega)$ and $\mathbf{c} \in L^\infty(\mathbf{div};\Omega)$. Here, $L^\infty(\mathbf{div};\Omega) = \big\{\mathbf{f} \in [L^\infty(\Omega)]^d \mid \nabla\cdot\mathbf{f} \in [L^\infty(\Omega)]\big\}$.*

Exercise 1.3. *Consider the problem:*
$$-\nabla\cdot\mathbf{A}(\mathbf{x})\nabla u + \mathbf{c}(\mathbf{x})\cdot\nabla u + b(\mathbf{x})u = f, \quad \Omega, \tag{1.5a}$$
$$u = 0, \quad \partial\Omega, \tag{1.5b}$$
where $\mathbf{A}$ and b satisfy the above assumptions (1.2) *and $\mathbf{c}(\mathbf{x})$ fulfills the following assumption:*
$$\frac{1}{2}\nabla\cdot\mathbf{c}(\mathbf{x}) \le b(\mathbf{x}) \quad a.e. \in \Omega.$$
Then, using the above Lax–Milgram Lemma, show that the above problem (1.5) *has a unique solution in $H_0^1(\Omega)$.*

1.1.4 Equivalence of three formulations

Introduce the quadratic functional $E : V \to \mathbf{R}$ given by
$$E(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle_{V',V} = \frac{1}{2}a(v,v) - \ell(v), \tag{1.6}$$
where E is called a complementary potential energy.

Consider the minimization problem:
$$\min_{v\in V} E(v), \tag{1.7}$$
or equivalently, find $u \in V$ such that
$$E(u) \le E(v) \quad \forall v \in V.$$
The following lemma is useful.

Lemma 1.2. *If w is continuous on Ω and*

$$\int_\Omega w\,\overline{v}\,dx = 0 \qquad \forall v \in H_0^1(\Omega).$$

Then $w(x) = 0$ for all $x \in \Omega$.

Proof. Exercise. Hint: use $\epsilon - \delta$ definition of continuous functions and $C_0^\infty(\Omega) \subset H_0^1(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. The support $\mathrm{supp}(v)$ of v means the set $\overline{\{x \in \Omega \mid v(x) \neq 0\}}$. Since Ω is a subset of $\mathbb{R}^d$, the compactness means $\mathrm{supp}(v)$ is bounded. □

Remark 1.1. *Recall elliptic regularity for higher dimensional elliptic problems: Let u be the solution of $a(u,v) = (f,v)$, $v \in V$, where*

$$a(v,w) = \sum_{i,j=1}^d \left(a_{ij}\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_j} \right) + (c\,v, w)$$

$$V = \{v \in H^1(\Omega) \,:\, v = 0 \quad \text{on } \partial\Omega\}$$

If Ω is a convex polygonal domain and $f \in L^2(\Omega)$, then the variational solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$\|u\|_{2,\Omega} \le C\,\|f\|_{0,\Omega} \qquad [Gri85] \tag{1.8}$$

If $\partial\Omega$ is smooth, and $f \in H^k(\Omega)$, then the variational solution $u \in H^{k+2}(\Omega) \cap H_0^1(\Omega)$, and

$$\|u\|_{k+2,\Omega} \le C(k,\Omega)\,\|f\|_{k,\Omega} \qquad \text{for } k = 0, 1, \cdots \tag{1.9}$$

Theorem 1.1. *Let $u, v \in H^1(\Omega)$ and $\mathbf{A} \in L^\infty(\Omega)$. Moreover, assume that $\nabla\cdot(\mathbf{A}\nabla u) \in L^2(\Omega)$. Then the normal trace $\boldsymbol{\nu}\cdot(\mathbf{A}\nabla u) \in H^{-1/2}(\partial\Omega)$, and the following integration by parts formula holds:*

$$(\mathbf{A}\nabla u, \nabla v) = -(\nabla\cdot(\mathbf{A}\nabla u), v) + \langle \boldsymbol{\nu}\cdot(\mathbf{A}\nabla u), v\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \qquad \forall v \in H^1(\Omega). \tag{1.10}$$

Theorem 1.2. *The three problems* (1.1), (1.4) *and* (1.7) *are equivalent under a sufficient regularity condition such as $\nabla\cdot(\mathbf{A}\nabla u) \in L^2(\Omega)$ and $f \in L^2(\Omega)$.*

Proof. **Step 1.** (1.1)⇒(1.4). This was already shown above.

Step 2. (1.4)⇒(1.1). Assume that $v \in V$ satisfies (1.4). Generally, this doesn't hold. But, if $\nabla\cdot A\nabla u \in L^2(\Omega)$, one can use the integration by parts Theorem 1.1. Thus,

$$\int_\Omega (\mathscr{L}u - f)\,\overline{v}\,dx = 0, \qquad \forall v \in V.$$

By the above Lemma 1.2,

$$\mathscr{L}u - f = 0 \quad \text{in } V'.$$

Step 3. (1.4)⇒(1.7). Suppose u is a solution of (1.4). Then, for an arbitrary $v \in V$, we have

$$\begin{aligned} E(v) - E(u) &= \frac{1}{2}a(v,v) - \ell(v) - \left[\frac{1}{2}a(u,u) - \ell(u)\right] \\ &= \frac{1}{2}(a(u,u) - 2a(u,v) + a(v,v)) \\ &= \frac{1}{2}a(u-v, u-v) \ge 0, \end{aligned}$$

where we used $a(u, u-v) = \ell(u-v)$, since u solves (1.4).

Step 4. (1.7)⇒(1.4). Suppose u is a solution of (1.7). Then for any $\epsilon \in \mathbf{R}$,

$$E(u) \le E(u+\epsilon v) \Longleftrightarrow g(\epsilon).$$

Since $g(\epsilon)$ has a minimum at $\epsilon = 0$, $g'(0) = 0$. Therefore,

$$\begin{aligned} g'(0) &= \frac{d}{d\epsilon}\left[\frac{1}{2}a(u+\epsilon v, u+\epsilon v) - \langle f, u+\varepsilon v\rangle_{V',V}\right]\Big|_{\epsilon=0} \\ &= a(u,v) - \langle f, v\rangle_{V'V} = 0. \end{aligned}$$

□

Theorem 1.3. *A solution to* (1.4) *is unique if it exists, under the stated assumptions.*

Proof. Suppose u_1 and $u_2 \in V$ solve (1.4). Then

$$a(u_1, v) = \langle f, v\rangle_{V',V}, \tag{1.11}$$

$$a(u_2, v) = \langle f, v\rangle_{V',V} \quad \text{for all } v \in V. \tag{1.12}$$

By subtracting, $\quad a(u_1 - u_2, v) = 0 \quad$ for all $v \in V$.
Choose $v = u_1 - u_2 \in V$. Then

$$a(u_1 - u_2, u_1 - u_2) = 0, \tag{1.13}$$

from which by the V–coercivity of a, $\|u_1 - u_2\|_V \le Ca(u_1 - u_2, u_1 - u_2) = 0$ Hence, $u_1(x) - u_2(x) \Longleftrightarrow$ constant for all $x \in \Omega$. Since $u_1(0) - u_2(0) = 0$, $u_1(x) - u_2(x) = 0$, for all $x \in \Omega$. □

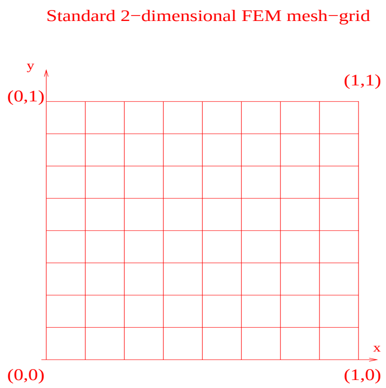
1.2 Finite Element Methods

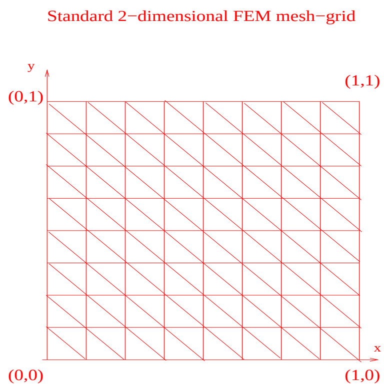
The finite element methods to solve (1.3) usually consist of the following procedure.

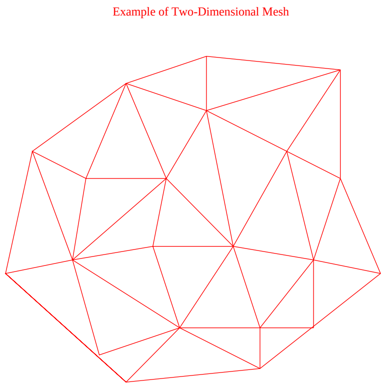
1. $(\mathscr{T}_h)_h$: Triangulation of the domain Ω
2. $(V_h)_h$: Construction of FE space: $V_h \to V$ as $h \to 0$
3. FE solution: Find $u_h \in V_h$ for each h
4. Error Analysis: Convergence of $u_h \to u$ as $h \to 0$
5. Numerical Simulation

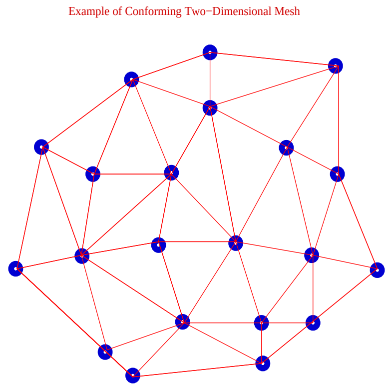
1.2.1 $\mathscr{T}_h$: triangulation of Ω

Decompose Ω into a union of finite number of elementary geometry



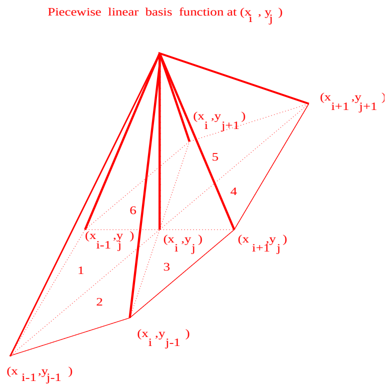


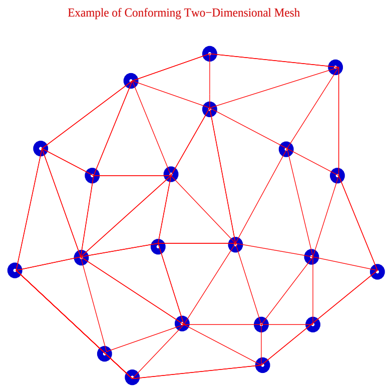




1.2.2 Construction of FE space

Based on $\mathscr{T}_h$ construct finite dimensional subspace V_h





- Let $V_h \subseteq V = H_0^1(\Omega)$ given by

$$V_h = \{v \in C^0(\Omega) : v|_K \text{ is piecewise linear for all } K \in \mathscr{T}_h\}$$

for triangular decomposition $\mathscr{T}_h$ or

$$V_h = \{v \in C^0(\Omega) : v|_K \text{ is piecewise bilinear for all } K \in \mathscr{T}_h\}$$

for quadrilateral decomposition $\mathscr{T}_h$

- In general, the function spaces are based on piecewise polynomials such that $v|_K \in \mathscr{P}_\ell(K)$ for some $\ell \geq 0$.

- Let $\varphi_j, j = 1 \cdots, N$, be the basis functions for V_h

$$\begin{aligned} V_h &= \text{Span}\{\varphi_j,\ j = 1, \cdots, N\} \\ &= \{v = \sum_{j=1}^{N} c_j \varphi_j \ : c_j \in \mathscr{F},\ j = 1, \cdots, N\} \end{aligned}$$

- Let $\psi_j, j = 1 \cdots, M$, be the basis functions for W_h

$$W_h = \{w = \sum_{j=1}^{M} c_j\psi_j : c_j \in \mathscr{F}, j = 1, \cdots, M\}$$

1.2.3 Compute FEM solution

1. Find $u_h \in V_h$ such that

$$a(u_h, v) = \ell(v) \quad \forall v \in W_h. \tag{1.14}$$

2. Find $u_h = \sum_{j=1}^{N} \alpha_j\varphi_j$ such that

$$\sum_{j=1}^{N} a(\varphi_j, v)\alpha_j = \ell(v) \quad \forall v \in W_h.$$

3. Find $u_h = \sum_{j=1}^{N} \alpha_j\varphi_j$ such that

$$\sum_{j=1}^{N} a(\varphi_j, \varphi_k)\alpha_j = \ell(\varphi_k) \quad \forall \varphi_k \in W_h.$$

4. Denote $\mathbf{b} = (\ell(\varphi_1), \cdots, \ell(\varphi_M))^t$ $\quad \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N)^t$; $A_{kj} = a(\varphi_j, \varphi_k), k = 1, \cdots, M, j = 1, \cdots, N$
5. Solve the $M \times N$ linear system:

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}. \tag{1.15}$$

6. If $M > N$, $\boldsymbol{\alpha}$ minimizes

$$\|\mathbf{A}\boldsymbol{\alpha}' - \mathbf{b}\|$$

- V_h: solution space, W_h: test function space
- Galerkin method (Ritz method) if V_h=W_h in (2)
- Petrov-Galerkin method: if $V_h \neq W_h$ in (2)
- If either V_h or W_h is not a subspace of $H_0^1(\Omega)$, Nonconforming FEM.

Linear solvers

1. Direct method: Gaussian elimination (LU-decomposition) method, Cholesky decomposition method, QR-decomposition method
2. Iterative method: Gauss-Jacobi and Gauss-Seidel methods, SOR, Richardson iteration method, Gradient method, Conjugate Gradient Method, BiConjugate Gradient Method, Domain Decomposition Method, MultiGrid solver,....

1.2.4 Convergence study: Galerkin method

Error equation

Subtracting (1.3) from (1.14), the error equation holds:

$$a(u_h - u, \varphi) = 0 \quad \forall \varphi \in V_h. \tag{1.16}$$

Lemma 1.3 (Ceá's lemma)**.**

$$\|\mathbf{u}_h - \mathbf{u}\|_1 \le \frac{C}{\alpha} \inf_{\mathbf{v}\in V_h} \|\mathbf{v} - \mathbf{u}\|_1. \tag{1.17}$$

In particular, $v = P_h\mathbf{u}$*, "the* V_h*-interpolation of* $\mathbf{u}$*", leads to*

$$\|\mathbf{u}_h - \mathbf{u}\|_1 \le \frac{C}{\alpha}\|\mathbf{u} - P_h\mathbf{u}\|_1. \tag{1.18}$$

Proof. From (1.16), coercivity, and boundedness; for any $v \in V_h$,

$$\begin{aligned}
&\alpha\|\mathbf{u}_h - \mathbf{u}\|_1^2 \le a(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{u})\\
&= a(\mathbf{u}_h - \mathbf{v} + (\mathbf{v} - \mathbf{u}), \mathbf{u}_h - \mathbf{u}) = a(\mathbf{v} - \mathbf{u}, \mathbf{u}_h - \mathbf{u})\\
&\le C\|\mathbf{v} - \mathbf{u}\|_1\|\mathbf{u} - \mathbf{u}_h\|_1.
\end{aligned}$$

Therefore, by dividing by $\mathbf{u} - \mathbf{u}_h\|_1$, one gets (1.17). In particular, $v = P_h\mathbf{u}$, "the V_h-interpolation of $\mathbf{u}$", leads to (1.18) □

By the interpolation property $\|\mathbf{u} - P_h\mathbf{u}\|_1 \le C_1 h|\mathbf{u}|_2$,

$$\|\mathbf{u}_h - \mathbf{u}\|_1 \le C_1\frac{C}{\alpha}h|\mathbf{u}|_2.$$

Hence, FE error : ellipticity * approximation properties in the FE space

Duality argument for $L^2(\Omega)$ error estimate

The aim is to show

$$\|\mathbf{u}_h - \mathbf{u}\| \le Ch^2|\mathbf{u}|_2,$$

under elliptic regularity conditions.

In order to obtain an L^2-error estimate, the following inclusions and identifications are used:

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \Longleftrightarrow L^2(\Omega)' \hookrightarrow H^{-1}(\Omega).$$

If $\phi \in L^2(\Omega)$, obviously $\phi \in H^{-1}(\Omega)$. Then, for all $v \in H_0^1(\Omega)$, one has

$$\langle \phi, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (\phi, v)_{L^2(\Omega)}.$$

We now state and prove the following lemma:

Lemma 1.4 (Aubin-Nitsche Lemma). *Let H be Hilbert space with the norm $\|\cdot\|_H$ and V be a subspace of H which is a Hilbert space with the norm $\|\cdot\|_V$. Suppose $V \hookrightarrow H$ is continuous. Then for any subspace $V_h \subset V$ fulfilling*

$$a(u_h - u, v) = 0 \qquad v \in V_h, \tag{1.19}$$

the following estimation is valid:

$$\|u_h - u\|_H \le C\|u_h - u\|_V \sup_{g\in H} \frac{\inf_{v\in V_h} \|v - \varphi_g\|_V}{\|g\|_H},$$

where $\varphi_g \in V$ is the unique solution of

$$a(w, \varphi_g) = \langle g, w\rangle_{V',V} \qquad \forall w \in V, \tag{1.20}$$

where the bilinear and liner forms satisfy the assumptions in the Lax-Milgram Lemma.

Proof. Let $g \in H \Longleftrightarrow H' \hookrightarrow V'$. In (1.20), the choice of $w = u_h - u$ leads to for any $v \in V_h$, by (1.19)

$$\begin{aligned} |\langle g, u_h - u\rangle_{V',V}| &= |a(u_h - u, \varphi_g)| = |a(u_h - u, \varphi_g - v)| \\ &\le C\,\|u_h - u\|_V\,\|\varphi_g - v\|_V\,. \end{aligned}$$

Now, exploiting the Gelfand triplet: $(g, u_h - u)_H = \langle g, u_h - u\rangle_{V',V}$ for $g \in H$, one has

$$\|u_h - u\|_H = \sup_{g\in H} \frac{(g, u_h - u)_H}{\|g\|_H} \le C\,\|u_h - u\|_V \sup_{g\in H} \frac{\inf_{v\in V_h} \|\varphi_g - v\|_V}{\|g\|_H}.$$

This proves the lemma. □

Corollary 1.1. *Let $\mathscr{T}_h$ be shape-regular and $u \in V \Longleftrightarrow H_0^1(\Omega)$ be the solution of $a(u, v) = \langle f, v\rangle_{V',V}$ for every $v \in H_0^1(\Omega)$. Then,*

$$\|u_h - u\|_{L^2(\Omega)} \le Ch\,\|u_h - u\|_{H^1(\Omega)}\,.$$

Moreover, assume that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies $a(u, v) = \langle f, v\rangle_{V',V}$ for every $v \in H^2(\Omega)$, then

$$\|u_h - u\|_0 \le Ch^2\,|u|_2\,.$$

Proof. Set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ in the Aubin-Nitsche Lemma, with $\|\cdot\|_H = \|\cdot\|_0$ and $\|\cdot\|_V = \|\cdot\|_1$ respectively. Notice that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous. Thus by the Aubin-Nitsche Lemma and elliptic regularity,

$$\begin{aligned} \|u_h - u\|_0 &\le C\,\|u_h - u\|_1 \sup_{g\in L^2(\Omega)} \left(\frac{\inf_{v\in V_h} \|\varphi_g - v\|_1}{\|g\|_0} \right) \\ &\le Ch\,\|u_h - u\|_1 \;\le\; Ch^2 |u|_2. \end{aligned}$$

□

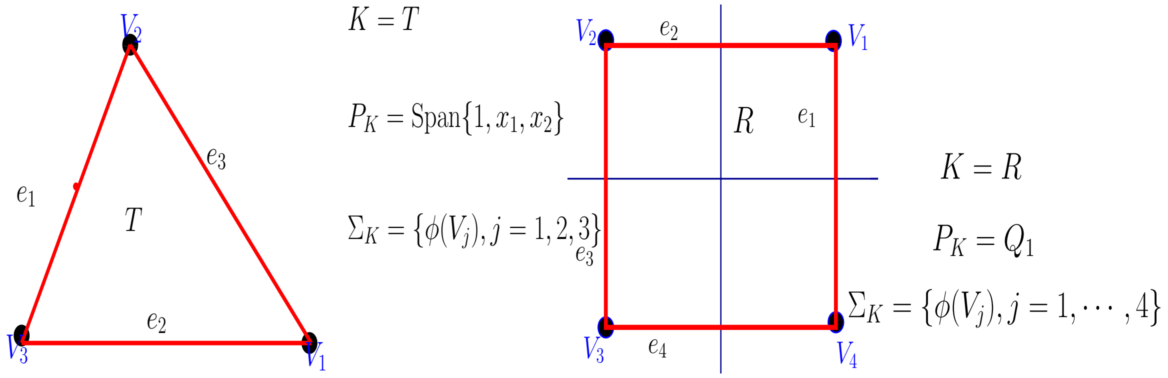
1.2.5 Finite elements

Finite element:$(K, \mathscr{P}_K, \Sigma_K)$

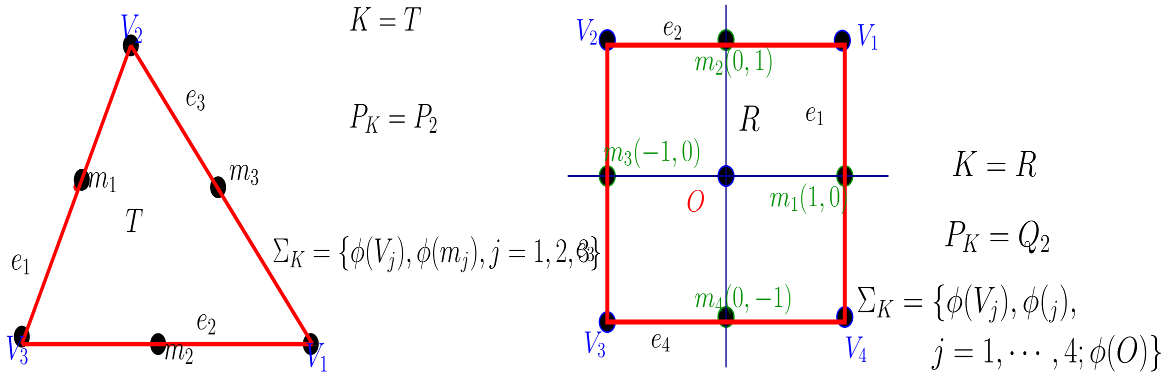
1. K: "element"–geometric object, triangle, quadrilateral, simplex, hexahedron, ...
2. $\mathscr{P}_K$: "local finite element space"–vector space of polynomials defined on K

3. Σ_K: "DOFs"–(Degrees of freedom) to determine an element $p \in P_K$ uniquely

- The Courant element or P_1 triangular element and the bilinear element or the Q_1 element.



- P_2 triangular element and Q_2-rectangular element



Here, and in what follows, the following notations for polynomial spaces are used:

$$P_k = \mathrm{Span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mid \alpha_1 + \cdots + \alpha_d \leq k, \alpha j \geq 0\}, \tag{1.21}$$

$$Q_k = \mathrm{Span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mid 0 \leq \alpha_j \leq k\}. \tag{1.22}$$

1.3 Construction of basis functions and calculation of matrix components

1.3.1 The 2–dimensional P_1 triangular conforming element (the Courant element) and the Crouzeix–Raviart P_1 nonconforming element

Consider a triangle $\overline{K}$ with vertices $\mathbf{V}_j = (p_j, q_j), j = 1, 2, 3$. Let $\mathbf{P}_j = (x_j, y_j), j = 1, 2, 3$, be three points on the triangle $\overline{K}$. We consider the two cases:

1. $\mathbf{P}_j = \mathbf{V}_j, j = 1, 2, 3$: the P_1 triangular conforming element
2. $\mathbf{P}_j = \mathbf{M}_j = \frac{\mathbf{V}_{j-1} + \mathbf{V}_j}{2}, j = 1, 2, 3$: the Crouzeix–Raviart P_1 nonconforming element

The basis function $\phi(j, x, y)$ associated with P_j is constructed as follows:

$$\phi(j, x, y) = \frac{(x - x_k)(y_k - y_l) - (x_k - y_l)(y - y_k)}{(x_j - x_k)(y_k - y_l) - (x_k - x_l)(y_j - y_k)}, \quad \{j, k, l\} = \{1, 2, 3\}. \tag{1.23}$$

Then obviously, $\phi(j, \mathbf{P}_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta. The gradient of $\phi(j,x,y)$ is given as follows:

$$\nabla\phi(j,x,y) = \frac{1}{(x_j - x_k)(y_k - y_l) - (x_k - y_l)(y_j - y_k)} \begin{pmatrix} y_k - y_l \\ -(x_k - y_l) \end{pmatrix}, \quad \{j,k,l\} = \{1,2,3\}. \tag{1.24}$$

The area of K is given by the cross product of vectors:

$$|K| = \frac{1}{2}\left|(\mathbf{V}_1 - \mathbf{V}_3) \times (\mathbf{V}_2 - \mathbf{V}_3)\right| = \frac{1}{2}|\mathbf{V}_1 - \mathbf{V}_3|\,|\mathbf{V}_2 - \mathbf{V}_3||\sin\theta| = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}\right|, \tag{1.25}$$

where θ is the angle between the two vectors.

1.3.2 Calculation of matrix components

Now let us compute the matrix component $A_{kj} = a(\phi_j, \phi_k)$ in (1.15).

$$\begin{aligned} A_{kj} &= a(\phi_j,\phi_k) = \int_\Omega (\mathbf{A}\nabla\phi_j)\cdot\nabla\phi_k + b(\mathbf{x})\phi_j\phi_k \, d\mathbf{x} \\ &= \sum_{K\in\mathscr{T}_h} \int_K (\mathbf{A}(\mathbf{x})\nabla\phi_j)\cdot\nabla\phi_k + b(\mathbf{x})\phi_j\phi_k \, d\mathbf{x} \\ &=: \sum_{K\in\mathscr{T}_h} a^K(\phi_j,\phi_k) =: \sum_{K\in\mathscr{T}_h} A_{kj}^K. \end{aligned}$$

Exercise 1.4. *(Due the start of class of July 3) Submit the program code with input data, if there is any, and analysis.*

Consider the elliptic equation (1.1) with discontinuous coefficient. The domain is taken as $\Omega = (0,1)^2$ *with* $\Omega_- = \left(\frac{1}{4}, \frac{3}{4}\right)^2$ *and the coefficients* $\mathbf{A}(\mathbf{x}) = a\chi_{\Omega_-}(\mathbf{x})\mathbf{I} + \chi_{\Omega\setminus\Omega_-}(\mathbf{x})\mathbf{I}$ *and* $b(\mathbf{x}) = 1$ *for all* $\mathbf{x}\in\Omega$. *Here,* χ *is the characteristic function and* $\mathbf{I}$ *denotes the identity matrix. Define*

$$\phi(x) = \begin{cases} x - x^2, & x \in [0,\frac{1}{4}] \cup [\frac{3}{4},1], \\ \frac{3}{16} + \frac{1}{a}\left[\frac{1}{16} - (x-\frac{1}{2})^2\right], & x \in [\frac{1}{4},\frac{3}{4}]. \end{cases}$$

Then set $f(x,y) = -\nabla\cdot(\mathbf{A}(x,y)\nabla(\phi(x)\phi(y)) + \phi(x)\phi(y)$ *Try with* $a = \frac{1}{100}$. *Divide* Ω *into* $N_x \times N_y$ *uniform rectangles and then divide each rectangle into two triangles with the diagonal from the top left to the bottom right. Try with* $N_x \times N_y = 50\times 50, 100\times 100, 200\times 200$. *In each case, calculate the* $L^2(\Omega)$*–error*

$$\sqrt{\int_\Omega |u_h(x,y) - u(x,y)|^2\, dxdy}, \quad h = 1/50, 1/100, 1/200.$$

The linear system can be solved by using the multi-grid method or the Conjugate Gradient Method.

Let $u(x,y) = U(r,\theta)$. Then we have

$$\begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial U}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \tag{1.26}$$

Hence,

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial U}{\partial\theta} \end{pmatrix}. \tag{1.27}$$

Thus,

$$\Delta u = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{pmatrix} \right] \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial U}{\partial \theta} \end{pmatrix} \tag{1.28}$$

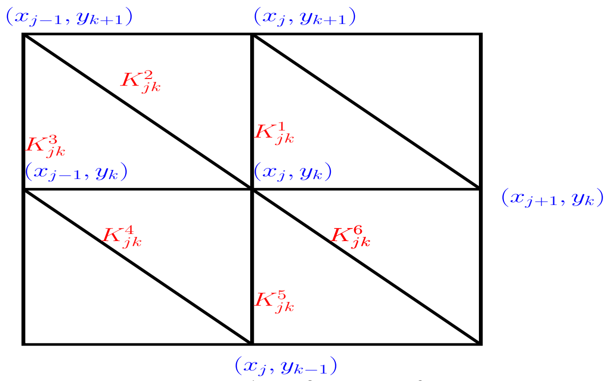
Or,

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta \frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial \theta} \\ \sin\theta \frac{\partial U}{\partial r} + \frac{\cos\theta}{r}\frac{\partial U}{\partial \theta} \end{pmatrix} \tag{1.29}$$

Consequently,

$$\begin{aligned} \Delta u = &= \left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\cos\theta\frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial\theta}\right] + \left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\sin\theta\frac{\partial U}{\partial r} + \frac{\cos\theta}{r}\frac{\partial U}{\partial\theta}\right] && (1.30) \\ &= \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}. && (1.31) \end{aligned}$$

1.3.3 Implementation of P_1 conforming triangular element on a uniform grid



We begin with the following simplest example:

$$\begin{aligned} -\Delta u &= f, \quad \Omega = (0,1)^2, && (1.32a) \\ u &= 0, \quad \partial\Omega. && (1.32b) \end{aligned}$$

Let $\mathscr{T}_h$ be the triangulation of Ω into the $2N \times 2N$ uniform triangles with vertices $V_{jk} = (x_j, y_k) = h(j,k), j,k = 0,1,\cdots,N$, with $h = \frac{1}{N}$.

In each triangle $K^1_{jk}, K^2_{jk}, \cdots, K^6_{jk}$ which has V_{jk} as a vertex, we have

$$\varphi_{jk}(x,y) = \begin{cases} -\frac{1}{h}(x+y-(j+k+1)h) & \text{on } K^1_{jk}, \\ -\frac{1}{h}(y-(k+1)h) & \text{on } K^2_{jk}, \\ \frac{1}{h}(x-(j-1)h) & \text{on } K^3_{jk}, \\ \frac{1}{h}(x+y-(j+k-1)h) & \text{on } K^4_{jk}, \\ \frac{1}{h}(y-(k-1)h) & \text{on } K^5_{jk}, \\ -\frac{1}{h}(x-(j+1)h) & \text{on } K^6_{jk}. \end{cases}$$

Let $u_h(x,y) = \sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \alpha_{jk}\varphi_{jk}(x,y)$ be an element of V_h.
Find α_{jk} for $j = 1,\cdots,N-1$ and $k = 1,\cdots,N-1$ such that

$$\sum_{j,k=1}^{N-1} \alpha_{jk} \int_\Omega \nabla\varphi_{jk}\cdot\nabla\varphi_{lm}dxdy = \int_\Omega f\,\varphi_{lm}dxdy$$

for all $l = 1,\cdots,N-1$, $m = 1,\cdots,N-1$.

Let

$$A_{lm,jk} = \int_\Omega \nabla\varphi_{jk}\cdot\nabla\varphi_{lm}\,dxdy \quad \text{and } b_{lm} = \int_\Omega f\,\varphi_{lm}\,dxdy.$$

If we write α (and b) as the following order:

$$(\alpha_{11}, \alpha_{21}, \cdots, x_{N-1,1}, \alpha_{12}, \alpha_{22}, \cdots, \alpha_{N-1,2}, \cdots, \alpha_{1,N-1}, \alpha_{2,N-1}, \cdots, \alpha_{N-1,N-1})^t,$$

then the matrix A satisfying $A\alpha = b$ will be of the form:

$$\begin{pmatrix} A_{11,11} & A_{11,21} & \cdots & A_{11,N-1,1} & \cdots & A_{11,1,N-1} & \cdots & A_{11,N-1,N-1} \\ A_{21,11} & A_{21,21} & \cdots & A_{21,N-1,1} & \cdots & A_{21,1,N-1} & \cdots & A_{21,N-1,N-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{N-1,1,11} & A_{N-1,1,21} & \cdots & A_{N-1,1,N-1,1} & \cdots & A_{N-1,1,1,N-1} & \cdots & A_{N-1,1,N-1,N-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1,N-1,11} & A_{1,N-1,21} & \cdots & A_{1,N-1,N-1,1} & \cdots & A_{1,N-1,1,N-1} & \cdots & A_{1,N-1,N-1,N-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{N-1,N-1,11} & A_{N-1,N-1,21} & \cdots & A_{N-1,N-1,N-1,1} & \cdots & A_{N-1,N-1,1,N-1} & \cdots & A_{N-1,N-1,N-1,N-1} \end{pmatrix}.$$

In the above matrix, the nonzero components may be found only if $|j-l|+|k-m| \le 1$ or $|j-l| = |k-m| = 1$.

To compute these nonzero terms, notice that $\nabla\varphi_{jk}$ in each region:

$$\nabla\varphi_{jk} = \begin{cases} \frac{1}{h}(-1,1) & \text{on } K_{jk}^1, \\ \frac{1}{h}(0,-1) & \text{on } K_{jk}^2, \\ \frac{1}{h}(1,0) & \text{on } K_{jk}^3, \\ \frac{1}{h}(1,-1) & \text{on } K_{jk}^4, \\ \frac{1}{h}(0,1) & \text{on } K_{jk}^5, \\ \frac{1}{h}(-1,0) & \text{on } K_{jk}^6. \end{cases}$$

1. Then $A_{jk,jk} = \sum_{j=1}^6 \int_{K_{jk}^j} |\nabla\varphi_{jk}|^2\,dxdy = [(\frac{1}{h})^2 + (\frac{1}{h})^2 + 2(\frac{1}{h})^2 + (\frac{1}{h})^2 + (\frac{1}{h})^2 + 2(\frac{1}{h})^2] \times \frac{h^2}{2} = 4.$
2. For $(l,m) = (j-1,k)$ [or $(j+1,k)$], we can compute $A_{lm,jk}$ by calculating integrals only on K_{jk}^3 and K_{jk}^4 [or K_{jk}^1 and K_{jk}^6]. Therefore, $A_{lm,jk} = -(\frac{1}{h})^2 \cdot \frac{h^2}{2} \times 2 = -1$.
3. Similarly, for $(l,m) = (j,k-1)$ [or $(j,k+1)$], we can compute $A_{lm,jk}$ by calculating integrals only on K_{jk}^5 and K_{jk}^1 [or K_{jk}^2 and K_{jk}^4]. Therefore, $A_{lm,jk} = -(\frac{1}{h})^2 \cdot \frac{h^2}{2} \times 2 = -1$.
4. Finally, for $(l,m) = (j-1,k-1)$ [or $(j+1,k+1)$], we can compute $A_{lm,jk}$ by calculating integral only on K_{jk}^3 and K_{jk}^5 [or K_{jk}^6 and K_{jk}^2]. In these cases, the gradients of φ_{jk} and φ_{lm} are perpendicular, so $A_{lm,jk} = 0$.

Combining the above calculations, we can see that

$$A = \begin{pmatrix} B & -I & \cdots & O \\ -I & B & \ddots & O \\ \vdots & \ddots & \ddots & \vdots \\ O & O & \cdots & B \end{pmatrix} \quad \text{where} \quad B = \begin{pmatrix} 4 & -1 & \cdots & 0 \\ -1 & 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 4 \end{pmatrix},$$

and **I** is the $(N-1)\times(N-1)$ identity matrix.

Similarly, b_{lm} can be calculated by

$$\int_\Omega \varphi_{lm} f\,dxdy = \sum_{jk} \int_{K_{jk}} \varphi_{lm}(x,y)f(x,y)\,dxdy. = \sum_{j=1}^6 \int_{K_{lm}^j} \varphi_{lm}(x,y)f(x,y)\,dxdy.$$

The jk-th row of $A\alpha = b$ is given by

$$4\alpha_{jk} - (\alpha_{j,k+1} + \alpha_{j,k-1} + \alpha_{j-1,k} + \alpha_{j+1,k}) = \int_\Omega \varphi_{jk} f\,dxdy.$$

For the constant force term, say, $f(x,y) = 1$ for all $(x,y) \in \Omega$, one can calculate $b_{jk} = \int_\Omega f(x,y)\varphi_{jk}(x,y)\,dxdy$ as the sum of volumes of six tetrahedrons:

$$b_{jk} = 6 \cdot \frac{h^2}{6} = h^2.$$

Notice that this is identical to the linear system where finite difference method applied, by dividing both sides by h^2..

In particular, if $f = 0$, then

$$\alpha_{jk} = \frac{1}{4}(\alpha_{j,k+1} + \alpha_{j,k-1} + \alpha_{j-1,k} + \alpha_{j+1,k}),$$

which is called the Discrete Mean Value Property.

Generally, if u is harmonic (i.e. $\Delta u = 0$), then u has the Mean Value Property:

$$u(x_0) = \frac{1}{2\pi r}\int_{S(x_0;r)} u(s)ds = \frac{1}{\pi r^2}\int_{B(x_0;r)} u(x)dx$$

where $S(x_0;r)$ and $B(x_0;r)$ denote circle and disk of radius r centered at x_0.

1.4 Higher order finite elements in $\mathbb{R}^d$

1.4.1 P_m (conforming) simplicial element

Let K be a non-degenerate simplex with vertices $V_1, \cdots, V_d, V_{d+1}$ with identification $V_0 = V_{d+1}$. Denote by $\mathscr{J}$ the set of indices $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_{d+1})$ in $\mathbb{Z}_+^{d+1}$ with $\sum_{j=1}^{d+1} \alpha_j = m$ Then consider the set of points

$$\mathbb{P} = \sum_{\alpha \in \mathscr{J}} \frac{\alpha_j}{m} V_j.$$

- $K = d-$simplex
- $P_K = P_m(K)$ with $\dim(P_K) = \binom{d+m}{m}$
- $\Sigma_K^P = \{\phi(V) : V \in \mathbb{P}\}$ as the degrees of freedom

1.4.2 Q_m (conforming) d-linear element

Similar as simplicial elements.

Let K be the d–cube $(-1,1)^d$. Then consider the set of points

$$\mathbb{P} = \left\{ \frac{2}{m}(\alpha_1, \cdots, \alpha_d) - (1, \cdots, 1) \in \mathbb{R}^d \mid \alpha_j \in \{0, 1, \cdots, m\}\, \forall j \right\}$$

- $K = d-$cube $= (-1,1)^d$
- $P_K = Q_m(K)$ with $\dim(P_K) = (m+1)^d$
- $\Sigma_K^P = \{\phi(V) : V \in \mathbb{P}\}$ as the degrees of freedom.

1.4.3 Unisolvency and optimality

It is necessary to show that the DOFs determine the function in P_K. Also in order to construct optimal finite elements, one usually require that $P_m(K) \subset P_K$. In this case the Bramble–Hilbert lemma shows the element is optimal. We state the following results without proof at the moment.

Theorem 1.4. *All the finite elements defined above have the "unisolvency" property. That is, if the DOFs of a given function $f \in P_K$ are zero, then the function $f \Longleftrightarrow 0$ in K.*

Also we have the continuity property as follows:

Theorem 1.5. *All the finite elements defined above have the "continuity" property. That is, if the DOFs restricted on a face f of K of a given function $f \in P_K$ are zero, then the function $f \Longleftrightarrow 0$ on the face f.*

1.4.4 Triangulations and reference elements

Let $(\mathscr{T}_h)_{0<h\le 1}$ be a family of triangulations of $\Omega \in \mathbb{R}^d$ into disjoint d–simplices or d-rectangles such that

$$\overline{\Omega} = \cup_{K\in\mathscr{T}_h} \overline{K}, \quad \overline{K} \cap \overline{K}' \text{ is a common face of } K \text{ and } K' \text{ whenever } \overline{K}\cap\overline{K}' \neq \emptyset.$$

For each $K \in \mathscr{T}_h$, denote by h_K the maximum diameter of K and by ρ_K the radius of balls inscribed in K. If there exists $\kappa > 0$ such that for every h,

$$\frac{h_K}{\rho_K} \le \kappa \text{ for all } K \in \mathscr{T}_h,$$

the family of triangulations is called "shape-regular". If, in addition, there exists $c > 0$ such that

$$h_K \ge ch \quad \forall K \in \mathscr{T}_h \quad \forall h,$$

the family of triangulations is called "quasi-uniform".

We now introduce the following reference elements:

1. d-simplicial element: Let $\widehat{K}$ be the reference simplicial element in $\mathbb{R}^d$ with coordinates $\widehat{V}_j, j = 0, 1, \cdot * d$ such that $\widehat{V}_0 = \mathbf{0}$ and $\widehat{V}_j - \widehat{V}_0 = \widehat{\mathbf{e}}_j$ be the j^{th} unit vector for $j = 1, \cdots, d$. Then, for each triangle or a simplex $K \in \mathscr{T}_h$, there is a surjective one–to–one affine map $\mathscr{F}_K : \widehat{K} \to K$ such that $\mathscr{F}_K(\widehat{\mathbf{x}}) = \mathbf{B}_K\widehat{\mathbf{x}} + \mathbf{b}_K$.

   In $\mathbb{R}^2$ and $\mathbb{R}^2$ the elements are called triangular and simplicial elements, respectively.

2. d-rectangle-type element: Let $\widehat{K}$ be the d–cube $(-1, 1)^d$. Then, for any d-rectangle $K \in \mathscr{T}_h$, there is a surjective one–to–one d-linear map $\mathscr{F}_K : \widehat{K} \to K$ such that $\mathscr{F}_K(\widehat{\mathbf{x}}) = \mathbf{B}_K(\widehat{\mathbf{x}})$, where each component of $\mathbf{B}_K : \widehat{K} \to K$ is a d-linear map.

   In $\mathbb{R}^2$ and $\mathbb{R}^2$ the elements are called quadrilateral and hexahedral elements, respectively. Notice that in two dimension the faces of a quadrilateral are straight lines segments, but in three dimension the faces of a hexahedron may not be flat.

On the reference element $\widehat{K}$ the reference finite element is denoted by the triple:

$$(\widehat{K}, \widehat{P}_{\widehat{K}}, \widehat{\Sigma}_{\widehat{K}}).$$

Remark 1.2. *The DOFs(degrees of freedom), denoted by $\widehat{\Sigma}_{\widehat{K}}$ or Σ_K, are evaluations of function $\widehat{\phi} \in \widehat{P}_{\widehat{K}}$ or $\phi \in P_K$. Hence the set $\widehat{\Sigma}_{\widehat{K}}$ or Σ_K which denotes the DOFs is a subset of dual space of $\widehat{P}_{\widehat{K}}$ or P_K, respectively. That is,*

$$\widehat{\Sigma}_{\widehat{K}} \subset (\widehat{P}_{\widehat{K}})' \textit{ with } \#(\widehat{\Sigma}_{\widehat{K}}) = \dim(\widehat{P}_{\widehat{K}}) \textit{ or } \Sigma_K \subset (P_K)' \textit{ with } \#(\Sigma_K) = \dim(P_K)$$

If the DOFs consist of the evaluation of function at points or the integrations of function on a subset of the element only, the finite element is called "Lagrange–type finite element". If they contain any evaluation of derivative of function or any integration of derivative, which is not included in the Lagrange–type DOFs, the finite element is called "Hermite–type finite element". All the finite elements introduced above are of Lagrange–type. There are a number of Hermite–type finite elements. We will introduce an important Hermite–type finite element, so–called the 21 DOFs "Argyris element" and the 6 DOFs "Morley nonconforming element"

Using the reference finite element, one can define the finite element (K, P_K, Σ_K) on each element $K \in \mathscr{T}_h$ as follows:

- $K = \mathscr{F}_K(\widehat{K})$
- $P_K = \{\widehat{p} \circ \mathscr{F}_K^{-1} \mid \widehat{p} \in \widehat{P}_{\widehat{K}}\}$
- $\Sigma_K = \{f \in (P_K)' \mid \Sigma(\phi) = \widehat{\Sigma}(\mathscr{F}_K), \quad \widehat{\Sigma} \in \widehat{\Sigma}_{\widehat{K}} \quad \forall \phi \in P_K\}$

Notice that in Σ_K there are at least three types of DOFs to distinguish as follows: with $\phi = \widehat{\phi} \circ \mathscr{F}_K^{-1}$, which is equivalent to $\widehat{\phi} = \phi \circ \mathscr{F}_K$,

1. Point–value type: $\widehat{\Sigma}(\widehat{\phi}) = \widehat{\phi}(\widehat{x}_j) \Longleftrightarrow \Sigma(\phi) = (\phi \circ \mathscr{F}_K)(\widehat{x}_j)$
2. Integral type: $\widehat{\Sigma}(\widehat{\phi}) = \int_{\widehat{e}} \widehat{\phi}(\widehat{x})\, d\widehat{\sigma}(\widehat{x}) \Longleftrightarrow \Sigma(\phi) = \int_e (\phi \circ \mathscr{F}_K^{-1})(x) \left|\frac{\partial \widehat{x}}{\partial x}\right| d\sigma(x)$
3. Derivative type: $\widehat{\Sigma}(\widehat{\phi}) = \frac{\partial \widehat{\Sigma}}{\partial \widehat{x}_k}(\widehat{x}_j) \Longleftrightarrow \Sigma(\phi) = \sum_{\ell=1}^d \frac{\partial(\phi \circ \mathscr{F}_K)}{\partial x_\ell}(\widehat{x}_j) \frac{\partial x_\ell}{\partial \widehat{x}_k}$

Let $\{\widehat{\phi}_j \mid j = 1, \cdots, J\}$ be the standard basis functions on the reference element $\widehat{K}$. Indeed, they are constructed as follows: denote

$$\widehat{\Sigma}_{\widehat{K}} = \{\widehat{\Sigma}_j \in (\widehat{P}_{\widehat{K}})', \quad j = 1, \ldots, \dim(\widehat{P}_{\widehat{K}})\}.$$

Associated to each $\widehat{\Sigma}_j$, find $\widehat{\phi}_j \in \widehat{P}_{\widehat{K}}$ such that

$$\widehat{\Sigma}_k(\widehat{\phi}_j) = \delta_{jk}, \quad \delta_{jk} \text{being the Kronecker delta function.} \tag{1.33}$$

Hence, the standard finite element basis functions are given by

$$\widehat{\phi}_j,\ j = 1, \cdots, \dim(\widehat{P}_{\widehat{K}}). \tag{1.34}$$

The basis functions on each $K \in \mathscr{T}_h$ is defined by the pull–back

$$\{\phi_{K,j}(\mathbf{x}) = (\widehat{\phi}_j \circ \mathscr{F}_K^{-1})(\mathbf{x}) \mid j = 1, \cdots, J\}.$$

Then the global finite element space V_h can be defined by

$$V_h = \{v_h \in C^0(\Omega) \mid v_h|_K = \sum_{j=1}^J \alpha_{K,j}\phi_{K,j} \quad \forall K \in \mathscr{T}_h\}. \tag{1.35}$$

Notice that in order to meet the criterion that $v_h \in C^0(\Omega)$, one needs to have

$$\alpha_{K,j} = \alpha_{K',j'} \quad \forall j' \text{ such that } \mathbf{x}_j(\in K) = \mathbf{x}_{j'}(\in K').$$

Definition 1.1. *Consider a finite element (K, P_K, Σ_K) and the global finite element space V_h. Let ℓ be the maximum order of derivatives in the definition of the DOFs Σ_K. For each $K \in \mathscr{T}_h$, define the local P_K–interpolation operator $\mathscr{I}_K : C^\ell(\overline{K}) \to P_K$ such that*

$$\Sigma_\ell(\mathscr{I}_K v - v) = 0 \quad \forall \Sigma_\ell \in \Sigma_K \quad \forall v \in C^\ell(\overline{K}). \tag{1.36}$$

Then the global V_h–interpolation operator $\mathscr{I}_h : \Omega \to V_h$ is defined piecewisely such that

$$\mathscr{I}_h v|_K(\mathbf{x}) = \mathscr{I}_K v \quad \forall K \in \mathscr{T}_h.$$

$\mathscr{I}_h v$ and $\mathscr{I}_K v$ are called the global V_h– and local P_K–interpolants of v, respectively.

Let $S \subset \mathbb{R}^d$ be a bounded set with $\operatorname{diam}(S) = h$. Bramble and Hilbert use the following normalized norms on $W^{m,p}(S)$ and $W^{m,\infty}(S)$, which are equivalent to the usual Sobolev norms:

$$|||u|||_{m,p,S} = \left(\sum_{k=1}^{m} h^{kp-d} |u|^p_{k,p,S} \right)^{\frac{1}{p}} \tag{1.37}$$

$$|||u|||_{m,\infty,S} = \sum_{k=0}^{m} h^k |u|_{k,\infty,S}. \tag{1.38}$$

Theorem 1.6 (Bramble–Hilbert I.). *Let $Q = W^{k,p}(S)/P_{k-1}$ be a quotient space. Then the semi–norm $h^{-d/p}|u|_{k,p,S}$ is a norm on Q equivalent to the quotient norm*

$$\|[u]\|_Q = \inf_{v\in[u]} h^{-d/p}\|v\|_{k,p,S}, \quad [u] \text{ denotes the equivalent class.}$$

Furthermore, there exists $C > 0$ independent of h and u such that

$$h^{k-\frac{d}{p}}\|u\|_{k,p,S} \le \|[u]\|_Q \le Ch^{k-\frac{d}{p}}\|u\|_{k,p,S} \quad \forall u \in W^{k,p}(S).$$

Theorem 1.7 (Bramble–Hilbert II.). *Let S be as in the above theorem. Suppose that $L : W^{k,p}(S) \to \mathbb{R}$ be a bounded linear functional such that*

1. $|L(u)| \le C|||u|||_{k,p,S} \quad \forall u \in W^{k,p}(S)$ *for $C > 0$ independent of h and u,*
2. $P_{k-1}(S) \subset Ker(L)$.

Then there exists $C > 0$ such that

$$|Lv| \le C_1 h^{k-d/p}|u|_{k,p,S} \quad \forall u \in W^{k,p}(S) \text{ for } C_1 > 0 \text{ independent of } h \text{ and } u.$$

Theorem 1.8 (Bramble–Hilbert III.). *Let S be as in the above theorem. Suppose that $L : W^{k,p}(S) \to \mathbb{R}$ be a bounded linear functional such that*

1. $|L(u)| \le C|||u|||_{j,\infty,S} \quad \forall u \in C^j(S)$ *for $C > 0$ independent of h and u,*
2. $P_{k-1}(S) \subset Ker(L)$.

Then there exists $C_1 > 0$ such that, for all $\frac{1}{p} - \frac{k-j}{d} < 0$, such that $W^{k,p}(S) \hookrightarrow C^j(S)$ and

$$|Lv| \le C_1 h^{k-d/p}|u|_{k,p,S} \quad \forall u \in W^{k,p}(S) \text{ for } C_1 > 0 \text{ independent of } h \text{ and } u.$$

Theorem 1.9 (Bramble–Hilbert IV.). *Let S be as in the above theorem. Suppose $u \in C^s$, $s = [s]+\alpha \in (0,1]$, where $[s]$ denotes the Gauss integer which is the largest integer $\le s$. Suppose that $L : C^0(S) \to \mathbb{R}$ be a bounded linear functional such that*

1. $|L(u)| \le C|u|_{0,\infty,S} \quad \forall u \in C^0(S)$ *for $C > 0$ independent of h and u,*
2. $P_{k-1}(S) \subset Ker(L)$.

Then there exists $C_1 > 0$ such that, for all $\frac{1}{p} - \frac{k-j}{d} < 0$, such that $W^{k,p}(S) \hookrightarrow C^j(S)$ and

$$|Lv| \le C_1 h^s \sup_{\mathbf{x},\mathbf{y}\in S} \sum_{|\boldsymbol{\alpha}|=[s]} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\boldsymbol{\alpha}}} \quad 0 \le s < k \quad \forall u \in W^{k,p}(S) \text{ for } C_1 > 0 \text{ independent of } h \text{ and } u.$$

Applications of Bramble–Hilbert Theorems are as follows: In Theorem 1.7, suppose $K \in \mathscr{T}_h$ be of size diam $(K) = h$. Then, choose $k = 2$ and $P_K = P_1(K)$. Consider the difference beween the following standard local P_1–interpolant and a function :

$$Lu(\mathbf{x}) = \mathscr{I}_K u(\mathbf{x}) - u(\mathbf{x}) = \sum_{j=1}^{\#(\text{vertices of K})} u(\mathbf{x}_j)\phi_{K,j}(\mathbf{x}) - u(\mathbf{x})$$

Then, verify that

$$|Lu(\mathbf{x})| = |\mathscr{I}_K u(\mathbf{x}) - u(\mathbf{x})| \le C|||u|||_{2,p,K}.$$

Thus from Theorem 1.7 it follows that

$$|Lu(\mathbf{x})| \le C_1 h^{2-\frac{d}{p}} |u|_{2,p,K}.$$

Hence,

$$\|\mathscr{I}_K u - u\|_{0,K} = \left(\int_K |Lu(\mathbf{x})|^p \, d\mathbf{x}\right)^{\frac{1}{p}} \le C_1 h^{2-\frac{d}{p}} |u|_{2,p,K} |K|^{\frac{1}{p}} = C_1 h^2 |u|_{2,p,K}.$$

In the meanwhile, for the energy–norm estimate, set $w_j = \frac{\partial u}{\partial x_j}$ choose $k = 1$ and $P_K = P_0(K)$. Consider the difference between the P_0–interpolation (*i.e.*, the mean–average operator on K) and a function:

$$Lw_j(\mathbf{x}) = \frac{1}{|K|}\int_K w_j(\mathbf{x})\, d\mathbf{x} - w_j(\mathbf{x}).$$

Then, verify that

$$|Lw_j(\mathbf{x})| \le C|||w_j|||_{1,p,K}.$$

Thus from Theorem 1.7 it follows that

$$|Lw_j(\mathbf{x})| \le C_1 h^{1-\frac{d}{p}} |w_j|_{1,p,K}.$$

Hence,

$$|\mathscr{I}_K u - u|_{1,p,K} = \left(\sum_{j=1}^{d}\int_K |Lw_j(\mathbf{x})|^p \, d\mathbf{x}\right)^{\frac{1}{p}} \le C_1 h^{1-\frac{d}{p}} |u|_{2,p,K} |K|^{\frac{1}{p}} = C_1 h^1 |u|_{2,p,K}.$$

Definition 1.2. *For $m \ge 1$, the broken norms (or mesh-dependent norms) are defined as follows:*

$$\|v\|_{m,h} := \left(\sum_{T\in\mathscr{T}_h} \|v\|_{m,T}^2\right)^{\frac{1}{2}} \qquad \forall v \in V_h. \tag{1.39}$$

Summarizing the above, we have

Theorem 1.10. *Let $m = 2$. Suppose that the finite element spaces P_K contain P_1, as all the finite elements introduced above. Then,*

$$\begin{aligned} \|\mathscr{I}_h u - u\|_{0,h} &\le Ch^2 |u|_{2,\Omega}, \\ \|\mathscr{I}_h u - u\|_{1,h} &\le Ch |u|_{2,\Omega}, \end{aligned}$$

for all $u \in H^2(\Omega)$. Invoking the Ceá lemma, we have the following error estimates between the finite element solution and the exact equation:

$$\|u_h - u\|_{1,h} \le C\, \|\mathscr{I}_h u - u\|_{1,h} \le Ch|u|_{2,\Omega}.$$

Theorem 1.11. *Let $m \ge 2$, $\{\mathscr{T}_h\}_h$ be a family of shape-regular triangulations of Ω, and V_h is a finite element space such that $P_{m-1}(\widehat{K}) \subset \widehat{P}_{\widehat{K}}$. Then, there exists a constant $C > 0$ such that*

$$\|u - \mathscr{I}_h u\|_{k,h} \le Ch^{m-k}|u|_{m,\Omega} \qquad \forall u \in H^m(\Omega) \text{ for } 0 \le k \le \min\{1, m\}.$$

1.5 Assembly of matrix $A_{kj} = \sum_{K\in\mathscr{T}_h} a_h(\phi_j, \phi_k)$

For this topics see Long Chen's implementation of iFEM.

Long Chen's lecture notes

"http://www.math.uci.edu/ chenlong/226/Ch3FEMCode.pdf"

Chapter 2

NC (Nonconforming) finite elements

All the finite element spaces V_h introduced earlier are subspaces of $H^1(\Omega) \cap C^0(\Omega)$. This means that any $v_h \in V_h$ are continuous across the interfaces $\Gamma_{\ell m} := \partial K_\ell \cap \partial K_m$ for all elements K_ℓ and K_m in $\mathscr{T}_h$. Denote by $[[\cdot]]_{\Gamma_{\ell m}}$ the jump across the interface $\Gamma_{\ell m}$ defined by

$$[[u]]_{\Gamma_{\ell m}} = \gamma_{\Gamma_{\ell m}}(u|_{K_\ell}) - \gamma_{\Gamma_{\ell m}}(u|_{K_m}), \tag{2.1}$$

where $u|_{K_\ell}$ denotes the restriction of $u \in L^2(\Omega)$ to K_ℓ, and $\gamma_{\Gamma_{\ell m}} u_{K_\ell}$ the trace of u_{K_ℓ} onto $\Gamma_{\ell m}$. Recall that K_ℓ's are open sets.

Conforming finite elements require that

$$[[u]]_{\Gamma_{\ell m}}(x) = 0 \quad \forall x \in \Gamma_{\ell m}. \tag{2.2}$$

Breaking this continuity restriction would substantially widen the realm of finite element spaces. However, there should be certain rules to relate functions defined on neighboring elements K_ℓ and K_m. These are essentially the degrees of freedom.

Let us classify DOFs (the degrees of freedom) in two categories:

1. Interface DOFs: the DOFs that transfer the information on the element to neighboring element
2. Interior DOFs: the DOFs that determine information only on the element

Observe that conforming finite elements have interior DOFs such that (2.2) holds.

Remark 2.1. *DOFs are also classified into two kinds as follows:*

1. *Lagrange type of finite elements: DOFs consist of point values of a function (usual P_{m-1} or Q_{m-1} finite elements)*
2. *Hermitian type of finite elements: DOFs consist of point values of the derivatives of the function upto certain order (Morley element, for instance)*

Nonconforming finite elements may have two types of interface DOFs

1. Gauss points DOFs: $\Sigma_K^P = \{\phi(g_j) : g_j$ are Gauss points on the faces of $K\}$
2. Orthogonality DOFs: $\Sigma_K^P = \{\int_f \phi q \, d\sigma : q \in P_{m-1}(f), f$ is a face of $K\}$

which relax (2.2) as follows

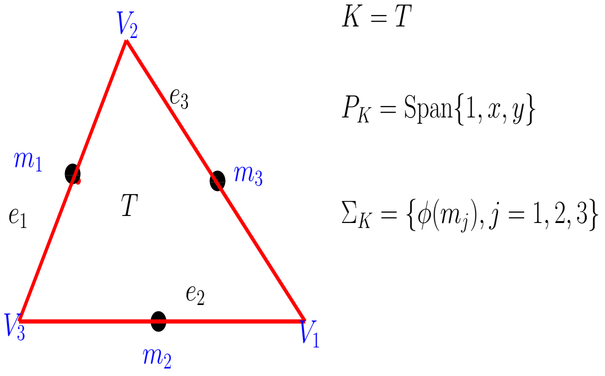
1. $[[u]]_{\Gamma_{\ell m}}(g_j) = 0 \quad \forall$Gauss points on the faces $\Gamma_{\ell m}$
2. $\int_{\Gamma_{\ell m}} [[u]]_{\Gamma_{\ell m}} q \, d\sigma = 0 \quad \forall q \in P_{m-1}(f), f$ is a face of $K\}$

respectively.

2.1 P_1 triangular NC element (Crouzeix-Raviart, 1973) and the rotated Q_1-rectangular NC elements

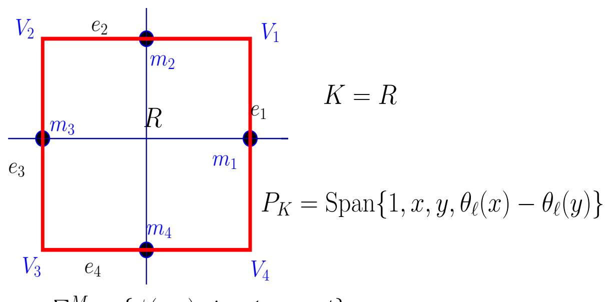
1. P_1 triangular NC element (Crouzeix-Raviart, 1973)

   - K = triangle
   - $P_K = \text{Span}\{1, x, y\}$
   - DOF: $\{\phi(m_j), m_j \text{midpoints of edges}, j = 1, 2, 3\}$



Exercise 2.1. *(Due to July 10) Let K be the triangle with vertices $(0,0),(1,0),(0,1),(0,0)$. Find three basis functions $\phi_j(x,y), j = 1, \cdots, 3$, for P_K such that $\phi_j(m_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta function.*

2. The rotated Q_1-rectangular NC elements (Han (1984), Rannacher–Turek (1992), Chen (1992), Douglas–Santos–Sheen–Ye (1999)



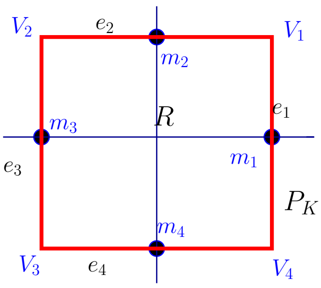
$\Sigma_K^M = \{\phi(m_j), j = 1, \cdots, 4\}$
$\Sigma_K^I = \{\int_{e_j} \phi \, d\sigma, j = 1, \cdots, 4\}$

   (a) H. Han (1984):
   - $K = R = [-1, 1]^2$
   - $P_K = \text{Span}\{1, x, y, x^2 - 5/3x^4, y^2 - 5/3y^4\}$
   - DOF: $\Sigma_K^M = \{\phi(m_j), m_j \text{midpoints of edges}, j = 1, 2, 3, 4\}$ plus $\int_K \phi$

   (b) Rannacher-Turek (1992, rotated Q_1 element, also Z. Chen):
   - $K = [-1, 1]^2$
   - $P_K = \text{Span}\{1, x, y, x^2 - y^2\}$

- DOF1 $\Sigma_K^P = \{\phi(m_j), m_j \text{midpoints of edges}, j = 1, 2, 3, 4\}$;
  DOF2: $\Sigma_K^I = \{\int_{e_j} \phi d\sigma, e_j \text{four edges}, j = 1, 2, 3, 4\}$;

(c) DSSY element(Douglas-Santos-Sheen-Ye, 1999): Cai-Douglas-Ye, 1999 stable Stokes element:



$K = R$

$P_K = \text{Span}\{1, x, y, \theta_\ell(x) - \theta_\ell(y)\}$

$\Sigma_K^M = \{\phi(m_j), j = 1, \cdots, 4\}$
$\Sigma_K^I = \{\int_{e_j} \phi \, d\sigma, j = 1, \cdots, 4\}$

- $K = [-1, 1]^2$
- $P_K = \text{Span}\{1, x, y, \theta_\ell(x) - \theta_\ell(y)\}, \ell = 0, 1, 2$
- DOF1 = DOF2; $\frac{1}{|e_j|} \int_{e_j} \phi \, d\sigma = \phi(m_j)$

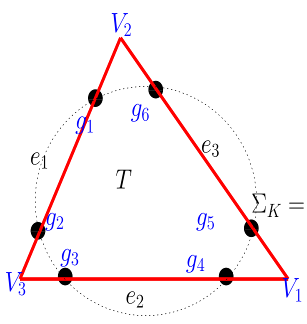
$$\theta_\ell(t) = \begin{cases} t^2, & \ell = 0; \\ t^2 - 5/3t^4, & \ell = 1; \\ t^2 - 25/6t^4 + 7/2t^6, & \ell = 2. \end{cases}$$

Exercise 2.2. *(Due to July 10) Let $K = (-1, 1)^2$. Find four basis functions $\phi_j(x, y), j = 1, \cdots, 4$, for P_K (DSSY element) such that $\phi_j(m_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta function.*

(d) For truly quadrilaterals, (Cai-Douglas-Santos-S.-Ye, CALCOLO, 2000):

- $K = [-1, 1]^2$
- $P_K = \text{Span}\{1, x, y, \theta_\ell(x) - \theta_\ell(y), xy\}, \ell = 0, 1, 2$
- DOF: $\{\phi(m_j), m_j \text{midpoints of edges}, j = 1, 2, 3, 4\}$ and $\int_{\widehat{R}} \phi(x, y) xy \, dxdy$;

3. P_2 triangular NC element (Fortin-Soulie, 1983; Lee-Sheen 2005) g_j's denotes Gauss points on the face

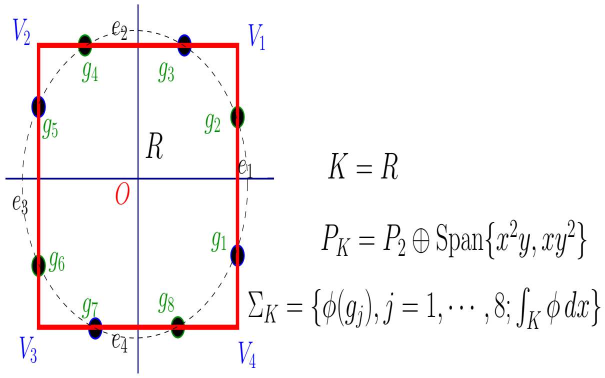


$K = T$

$P_K = P_2$

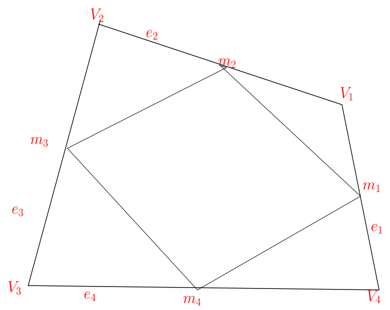
$\Sigma_K = \{\phi(g_j), j = 1, \cdots, 6; \int_K \phi \, dx\}$

4. P_2 rectangular NC element (or "rotated Q_2-rectangular NC element) (Lee-Sheen 2005) g_j's denotes Gauss points on the face



2.2 The P_1-NC quadrilateral and hexahedral elements

(C. Park, Thesis 2002, Park-Sheen SIAM J. Numer. Anal. 2003)



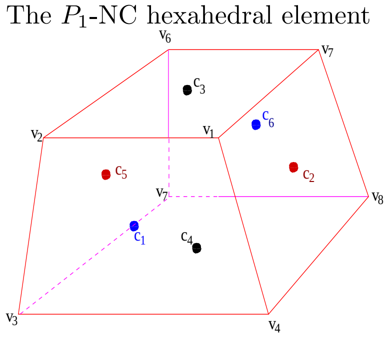
- $K = Q$: a quadrilateral
- $P_K = \mathscr{S}(Q) = \text{Span}\{1, x, y\}$
- $\Sigma_K = \{\phi_{M_j}, j = 1, \cdots, 4\}$
- Basis function

$$\phi_j(m) = \begin{cases} \frac{1}{2}, & \text{if } m = M_j, M_{j+1}, \\ 0, & \text{if } m = M_{j+2}, M_{j+3} \end{cases}$$

- $\mathscr{S}(Q) = \text{Span}\{\phi_1, \cdots, \phi_4\}$ and $dim(\mathscr{S}(Q)) = 3$

Lemma 2.1. *If $u \in S(Q)$, then $u(M_1) + u(M_3) = u(M_2) + u(M_4)$. Conversely, if u_j is a given value at M_j, for $j = 1, \cdots, 4$, satisfy $u_1 + u_3 = u_2 + u_4$, then there exists a unique function $u \in S(Q)$ such that $u(M_j) = u_j$, $j = 1, \cdots, 4$.*

The P_1-NC hexahedral element



- K = hexahedron
- $P_K = \mathscr{S}(Q) = \text{Span}\{1, x, y, z\}$
- $\Sigma_K = \{\phi(M_j), j = 1, \cdots, 6\}$
- $\mathscr{S}(Q) = \text{Span}\{\phi_1, \cdots, \phi_8\}$ and $dim(\mathscr{S}(Q)) = 4$

Lemma 2.2. *If $u \in S(Q)$, then $u(M_1) + u(M_6) = u(M_2) + u(M_5) = u(M_3) + u(M_4)$. Conversely, if u_j is a given value at M_j, for $1 \le j \le 6$, satisfying $u_1 + u_6 = u_2 + u_5 = u_3 + u_4$, then there exists a unique function $u \in S(Q)$ such that $u(M_j) = u_j$, $1 \le j \le 6$.*

With these elements we can prove optimal convergence.

Theorem 2.1. *The above element is unisolvent. Optimal error estimates holds for elliptic problems:*

2.2.1 Error analysis for linear nonconforming Galerkin method

Define the global nonconforming finite element space $\mathscr{N}C_0^h$ on $\mathscr{T}_h$ by

$$\begin{aligned}\mathscr{N}C_0^h &= \{v \in L^2(\Omega) \mid v|_K \in P_K \quad \forall K \in \mathscr{T}_h,\\ &\qquad \langle [[v]]_e, q\rangle_e = 0, \quad \forall q \in P_{m-1}(e), \langle v_e, q\rangle_e = 0, \quad \forall e \in \partial\}\end{aligned}$$

where $[[f]]_e$ means the jump of f across the interface e and $\langle [[v]]_e, q\rangle_e = \int_e [[v]]_e q\, d\sigma$.

Define the bilinear form $a_h : [\mathscr{N}C_0^h + H^1(\Omega)]^2 \to \mathbb{R}$ by

$$a_h(u,v) = \sum_{K\in\mathscr{T}_h} a_K(u,v), \qquad \text{where } a_K(u,v) \text{ denotes the integrations restricted to the domain } K$$

Define the *broken energy norm*: for $w \in H^1(\Omega) + \mathscr{N}C_0^h$

$$|w|_{1,h} = \sqrt{a_h(w,w)}$$

Recall that the weak solution $u \in H_0^1(\Omega)$ satisfies

$$a(u,v) = \langle f, v\rangle \quad \forall v \in H_0^1(O). \tag{2.3}$$

The nonconforming Galerkin method is to find $u_h \in \mathscr{N}C_0^h$ such that

$$a_h(u_h, v_h) = \langle f, v_h\rangle \quad \forall v_h \in \mathscr{N}C_0^h. \tag{2.4}$$

Lemma 2.3 (The second Strang Lemma). *If $u \in H_0^1(\Omega)$ and $u_h \in NC_0^h$ are the solutions of* (2.3) *and* (2.4) *respectively, then*

$$a_h(u-u_h, u-u_h) \le C\left(\inf_{v\in NC_0^h} |u-v|_{1,h} + \sup_{w\in NC_0^h} \frac{|a_h(u,w) - (f,w)|}{|w|_{1,h}}\right) \tag{2.5}$$

Remark 2.2. *In the right-hand side of* (2.5)*, the second term is called a consistency error term. Notice that in the consistency error term,*

$$a_h(u,w) - (f,w) = a_h(u,w) - a_h(u_h,w). \tag{2.6}$$

If $w \in C^0(\Omega)$, then the consistency error term become zero.

Proof. First, by the triangle inequality

$$|u-u_h|_{1,h} \le |u-z_h|_{1,h} + |z_h - u_h|_{1,h} \qquad \forall z_h \in NC_0^h.$$

Next, since $a_h(u_h, u_h - z_h) = (f, u_h - z_h)$, one has

$$\begin{aligned}|u_h - z_h|_{1,h}^2 &= a_h(u_h - z_h, u_h - z_h)\\ &= a_h(u - z_h, u_h - z_h) + a_h(u_h - u, u_h - z_h)\\ &= a_h(u - z_h, u_h - z_h) + (f, u_h - z_h) - a_h(u, u_h - z_h)\\ &\le |u - z_h|_{1,h}|u_h - z_h|_{1,h} + (f, u_h - z_h) - a_h(u, u_h - z_h).\end{aligned}$$

Then

$$|u_h - z_h|_{1,h} \le |u - z_h|_{1,h} + \frac{(f, u_h - z_h) - a_h(u, u_h - z_h)}{|u_h - z_h|_{1,h}},$$

which combined with the triangle inequality gives

$$|u - u_h|_{1,h} \le 2|u - z_h|_{1,h} + \frac{(f, u_h - z_h) - a_h(u, u_h - z_h)}{|u_h - z_h|_{1,h}}.$$

If we choose $z_h \in NC^h$ such that

$$|u - z_h|_{1,h} = \inf_{v \in NC_0^h} |u - v|_{1,h},$$

then

$$|u - u_h|_{1,h} \le 2 \inf_{v \in NC_0^h} |u - v|_{1,h} + \sup_{w \in NC_0^h} \frac{|(f,w) - a_h(u,w)|}{|w|_{1,h}}.$$

□

For the nonconforming finite elements introduced above P_K including $P_{m-1}(K), m = 2,3$, by the Bramble-Hilbert Theorem 1.7,

$$\|\mathscr{I}_h u - u\|_{m,\widehat{K}} \le C|u|_{m,\widehat{K}} \qquad \forall u \in H^m(\widehat{K}),$$

from which the first part in the second Strang lemma is estimated as follows:

$$\|\mathscr{I}_h u - u\|_{k,h} \le Ch^{m-k}|u|_{m,\Omega}, \qquad 0 \le k \le m. \tag{2.7}$$

2.2.2 Implementation of P_1–NC quadrilateral element

Consider the special case: $\Omega = (0,1)$ is decomposed into the uniform $N \times N$ uniform rectangles with coordinates $(x_j, y_k) = h(j,k), j,k = 0,1,\cdots,N$, where $h = \frac{1}{N}$. The basis functions $\phi_{jk}(x,y)$ contain four rectangles whose vertices contain (x_j,y_k). The basis function values are shown in the following, but in a shifted region $(-h,h)^2$.

To find components of the mass matrix, using symmetry and translation invariance, one has the following:

- $(l,m) = (j,k)$

$$(\phi_{jk}, \phi_{jk}) = \frac{4}{4h^2}\int_0^h\int_0^h \left(-x - y + \frac{3}{2}h\right)^2 dx\,dy = \frac{20}{48}h^2,$$

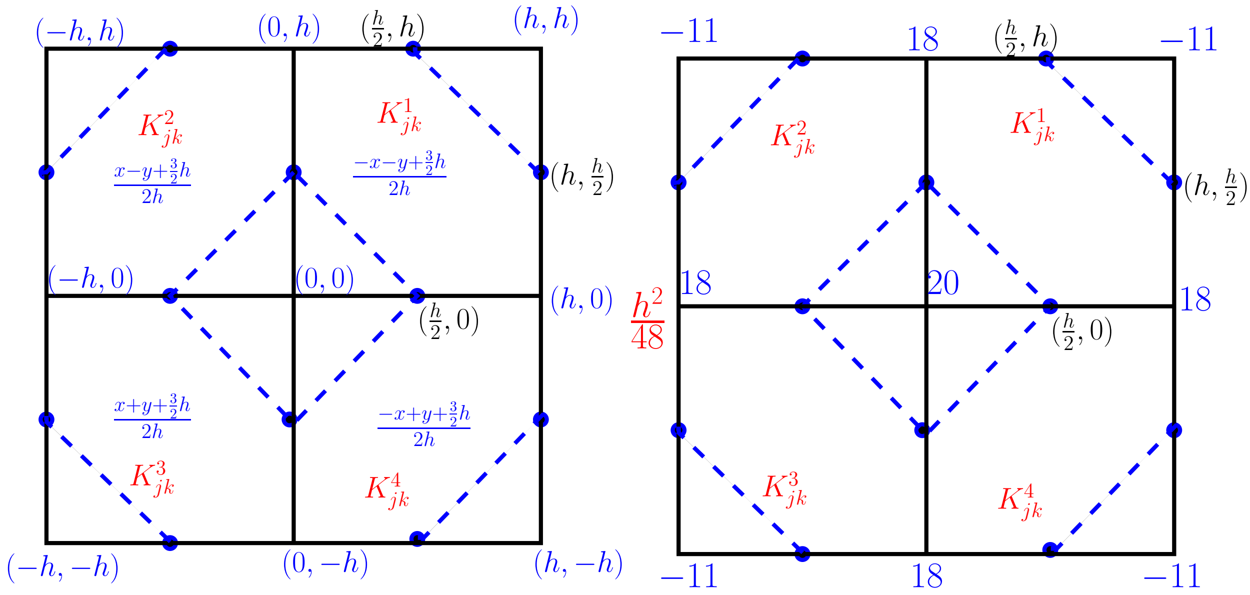
- $(l,m) = (j+1,k)$

$$(\phi_{jk}, \phi_{j+1,k}) = \frac{2}{4h^2}\int_0^h\int_0^h \left(\frac{-x-y+\frac{3}{2}h}{2h}\right)(x - y + \frac{h}{2})\,dx\,dy = \frac{18}{48}h^2,$$

and

- $(l,m) = (j+1,k+1)$

$$(\phi_{jk}, \phi_{j+1,k+1}) = \frac{1}{4h^2}\int_0^h\int_0^h \left(\frac{-x-y+\frac{3}{2}h}{2h}\right)(x + y - \frac{h}{2})\,dx\,dy = \frac{-11}{48}h^2.$$



$\phi(x, y)$ $\qquad$ $(\phi_{jk}, \phi_{\ell m})$

For the components of mass matrix, using symmetry and translation invariance,

- $(l, m) = (j, k)$

$$(\nabla\phi_{jk}, \nabla\phi_{jk}) = \frac{4}{4h^2}\int_0^h\int_0^h \begin{pmatrix}-1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\end{pmatrix}\,dx\,dy = 2,$$

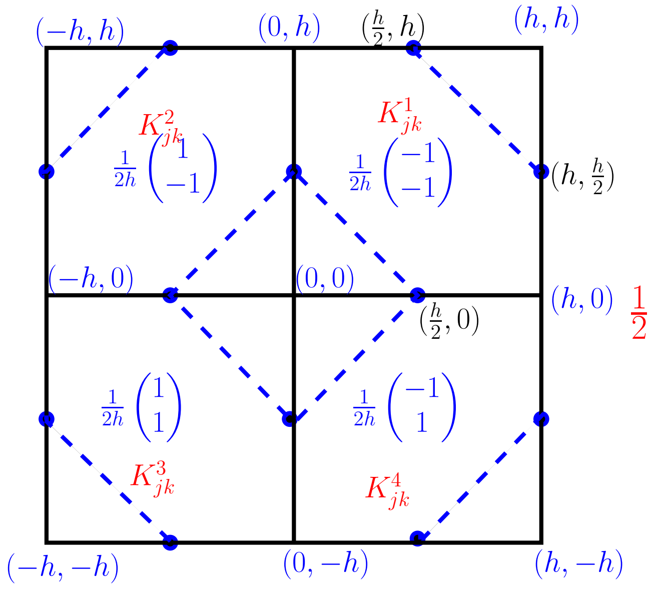
- $(l, m) = (j+1, k)$

$$(\nabla\phi_{jk}, \nabla\phi_{j+1,k}) = \frac{2}{4h^2}\int_0^h\int_0^h \begin{pmatrix}-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}\,dx\,dy = 0,$$

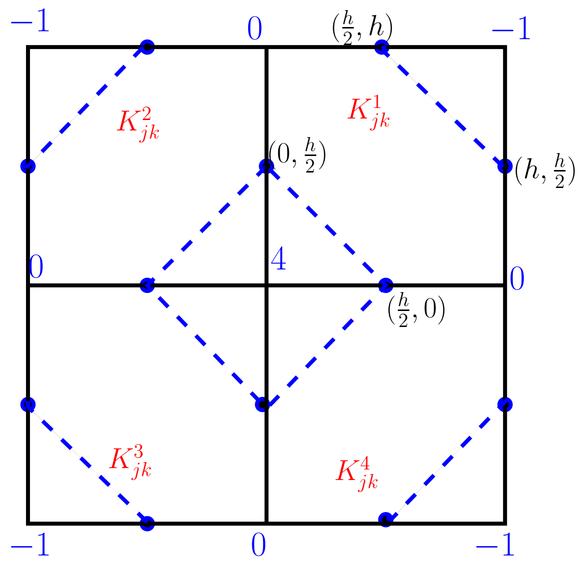
and

- $(l, m) = (j+1, k+1)$

$$(\nabla\phi_{jk}, \nabla\phi_{j+1,k+1}) = \frac{1}{4h^2}\int_0^h\int_0^h \begin{pmatrix}-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}\,dx\,dy = -\frac{1}{2}.$$



$\nabla\,\phi(x,y)$



$(\nabla\,\phi_{jk}, \nabla\,\phi_{\ell m})$

2.3 Hermite–type finite elements

In this section we introduce Hermite–type finite elements for solving the fourth–order partial differential equations (the biharmonic equation)

$$\begin{aligned} \Delta^2 u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.8}$$

Green's second identity:

$$\int_\Omega \Delta u\, v\, dx - \int_\Omega u\, \Delta v\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} d\sigma \tag{2.9}$$

By Green's second identity, we can have

$$(\Delta^2 u, v) = (f, v) \;\longrightarrow\; \int_\Omega ((\Delta^2 u) v\, dx - \int_\Omega \Delta u\, \Delta v\, dx = \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial \nu} v - \Delta u \frac{\partial v}{\partial \nu} d\sigma$$

$$\forall v \in H_0^2(\Omega) = \overline{C_0^\infty(\Omega)}^{H^2(\Omega)} \; (i.e.\ v|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0)$$

The weak problem is given as finding $u \in H_0^2(\Omega)$ such that

$$(\Delta u, \Delta v) = (f, v) \qquad \forall v \in H_0^2(\Omega) \tag{2.10}$$

Finite element method is to find $V_h \subset H_0^2(\Omega)$ and then to find $u_h \in V_h$ such that

$$(\Delta u_h, \Delta v) = (f, v) \qquad \forall v \in V_h \tag{2.11}$$

In particular, we want to find a finite element space $V_h \subset C^1(\overline{\Omega}) \cap H_0^2(\Omega)$.

Argyris element (C^1-quintic triangular element or 21-DOFs triangular element)

- T = triangle
- $P_T = P_5(T)$
- $\Sigma_T = \{\partial^\alpha p(a_i),\, |\alpha| \le 2,\, i = 1,2,3 \,:\, \partial_\nu p(a_{ij}.\, 1 \le i \le j \le 3\} = \Big\{ p(a_i),\ \frac{\partial p}{\partial x_1}(a_i),\ \frac{\partial p}{\partial x_2}(a_i),\ \frac{\partial^2 p}{\partial x_1^2}(a_i),\ \frac{\partial^2 p}{\partial x_1 \partial x_2}(a_i),\ \frac{\partial^2}{\partial x}$
  $i = 1,2,3\,;\ \frac{\partial p}{\partial \nu}(a_{ij}),\ 1 \le i \le j \le 3 \Big\}$
- $\dim(P_5(T)) = 21$.

Argyris triangle

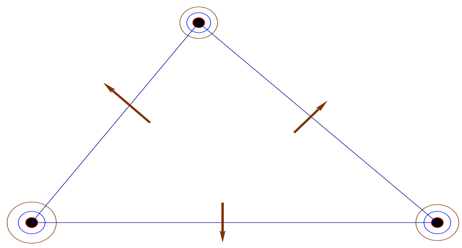


Figure 2.3.1: Argyris element

Let $v_h \in V_h$, where V_h is constructed using Argyris element. We want to show that

$$v_h|_{T_1} - v_h|_{T_2} = 0 \quad \text{along } \bar{T}_1 \cap \bar{T}_2.$$

Let t denotes the variable along $\bar{T}_1 \cap \bar{T}_2$ so that $v(t)$ is polynomial of degree ≤ 5 fulfilling

$v(t_1) = v(t_2) = v'(t_1) = v'(t_2) = v''(t_1) = v''(t_2) = 0 \Longrightarrow v(t)$ has factors $(t - t_1)^3$ and $(t - t_2)^3 \Longrightarrow v = 0$.

Of course, we have $v'(t) = 0$. Next, to show that the the normal directional derivative of v equals zero, let

$$w(t) = \left(\frac{\partial v_h}{\partial \nu}\Big|_{T_1} - \frac{\partial v_h}{\partial \nu}\Big|_{T_2} \right)(t),$$

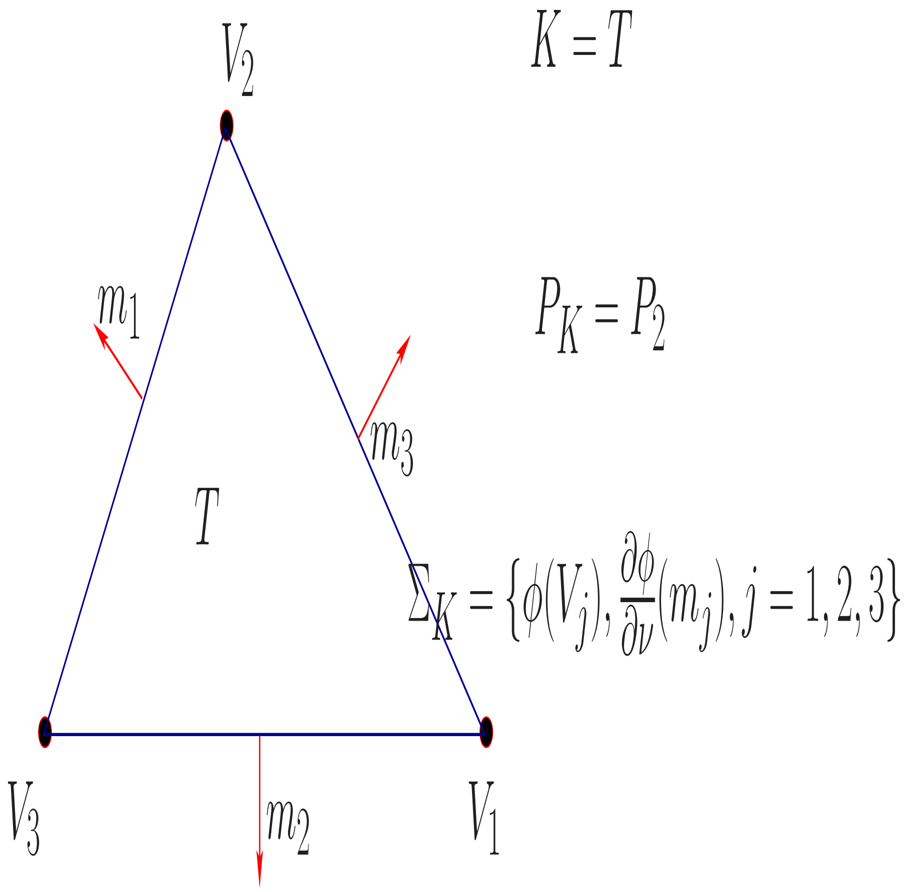
where ν denotes the normal vector from T_1 to T_2. Note that $w(t)$ is a polynomial of degree ≤ 4 and $w(t_1) = w(t_2) = w(t_{12}) = 0$, where t_{12} represents the midpoint between a_1 and a_2.

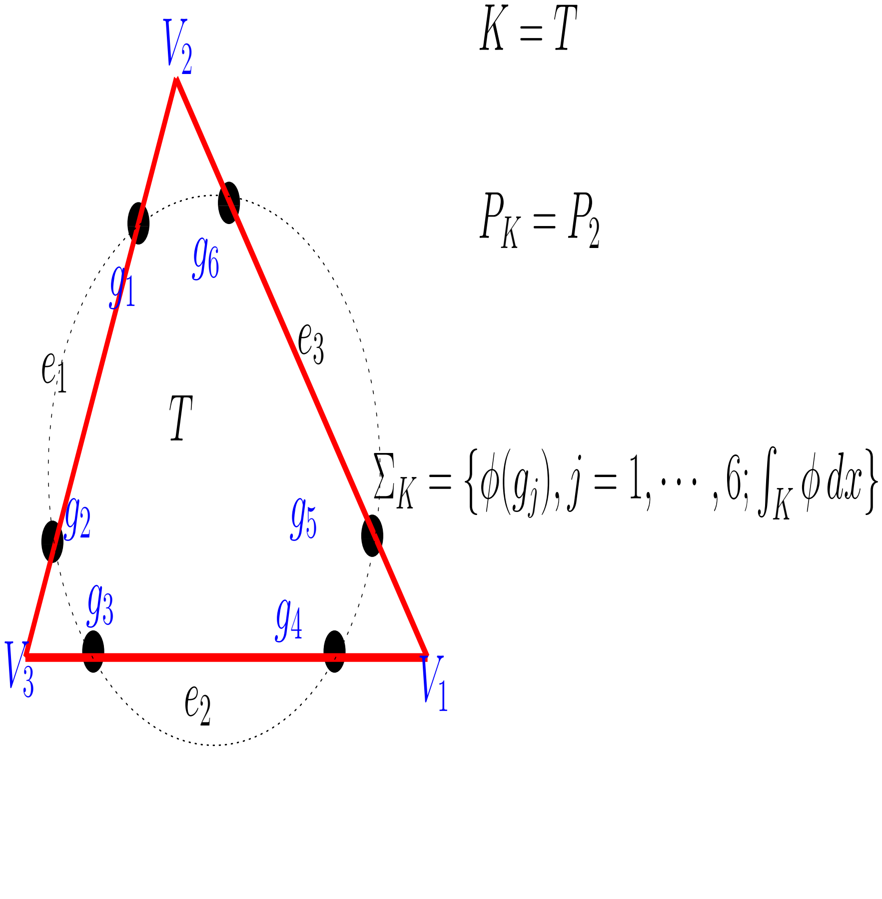
$$\begin{aligned} w'(t_j) &= \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \nu}\Big|_{T_1} - \frac{\partial v_h}{\partial \nu}\Big|_{T_2} \right)(t_j) \\ &= \nabla((\nabla v_h|_{T_1} \cdot \nu) - (\nabla v_h|_{T_2} \cdot \nu)) \cdot \tau = 0, \qquad j = 1,2. \end{aligned}$$

Thus $w(t) = 0$ along $\bar{T}_1 \cap \bar{T}_2$. And hence $v_h \in C^1(\overline{T}_1 \cup \overline{T}_2)$. This proves that $V_h \subset C^1(\overline{\Omega}) \cap H_0^2(\Omega)$.

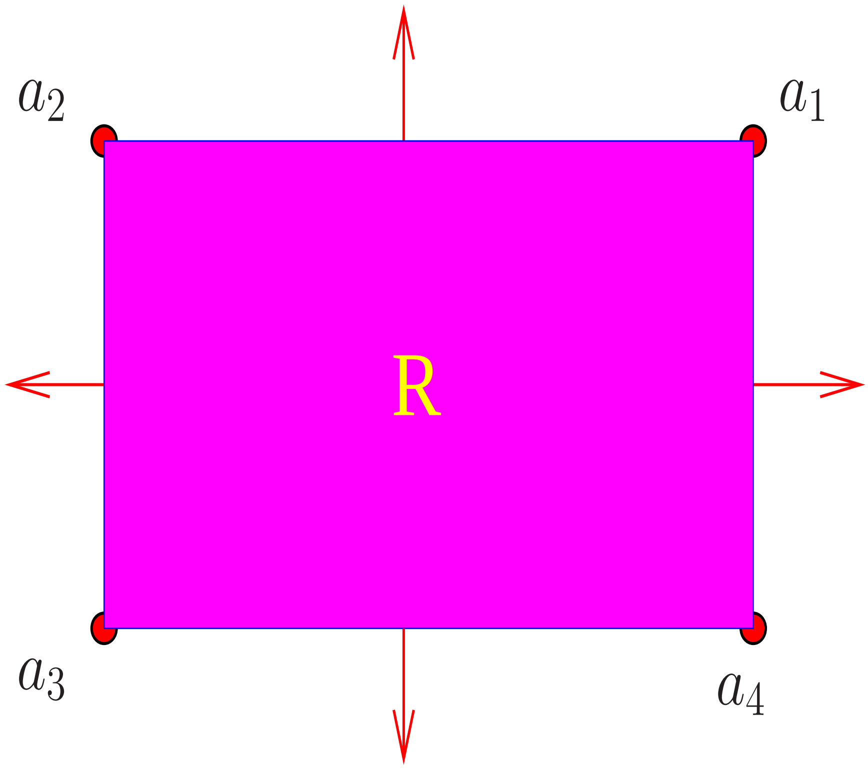
Quadratic nonconforming elements on rectangles with Heejeong Lee

1. Triangular case


V_2
$K = T$
m_1
$P_K = P_2$
m_3
T
$\Sigma_K = \{\phi(V_j), \frac{\partial \phi}{\partial \nu}(m_j), j = 1, 2, 3\}$
V_3
m_2
V_1


V_2
$K = T$
$P_K = P_2$
g_6
g_1
e_3
e_1
T
$\Sigma_K = \{\phi(g_j), j = 1, \cdots, 6; \int_K \phi\, dx\}$
g_2
g_5
g_3
g_4
V_3
V_1
e_2

Incomplete biquadratic element



- Morley element
- Fortin-Soulie element

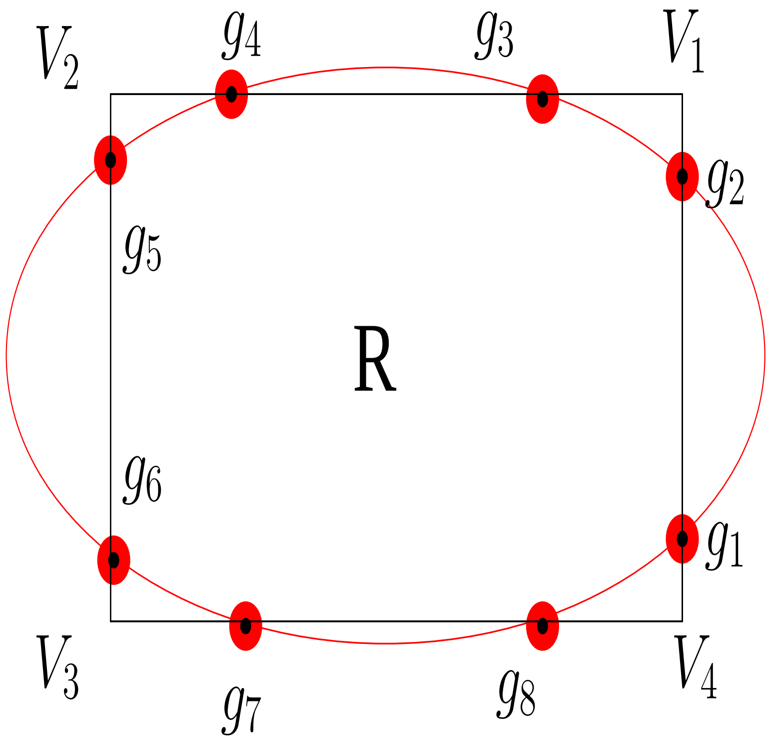
2. Rectangular case

- Incomplete biquadratic element
- Reduced incomplete biquadratic element (with Heejeong Lee)

Exercise 2.3 (Due July 24 (everything before the coding), and July 31 (coding))**.** *Choose any nonconforming finite element as you like. Then consider the same PDE as in the previous programming exercise.*

1. *Use* $N \times N$ *rectangular meshes with* $N = 50, 100, 200$.
2. *Find an explicit formula for the standard local basis functions on a reference domain. Using the reference basis functions, find an explicit formula for the global basis functions* ϕ'_{jk}*, where* ϕ_{jk} *is associated with the point* $(x_j, y_k) = h(j, k), h = \frac{1}{N}$.
3. *Give an explicit description how you would compute* $a_h(\phi_{jk}, \phi_{lm})$ *and* (f, ϕ_{jk}).
4. *Formulate a linear system* $Ax = b$*, where* x *denotes the unknown coefficients* $\alpha'_{jk}s$*. (Submit up to this by July 24.)*
5. *Use any numerical linear algebra to solve* $Ax = b$.

Quadratic nonconforming rectangle



6. *Compare the numerical solutions with the numerical solutions obtained by conforming finite elements.*
7. *Submit the codes and analysis and comments by July 31.*

Chapter 3

Preliminaries from functional analysis

Let Ω be a nonempty open set in $\mathbf{R}^d$, and set

- for $1 \le p < \infty$, $L^p(\Omega) = \{f : \Omega \to \mathbb{C} : \|f\|_{0,p,\Omega} < \infty\}$, where $\|f\|_{0,p,\Omega} := \left[\int_\Omega |f|^p dx\right]^{(1/p)}$,
- $C^k(\Omega) = \{f : \Omega \to \mathbb{C} : \text{all the derivatives of } f \text{ up to order } k \text{ are continuous}\}$,
- $C_0^k(\Omega) = \{f \in C^\kappa(\Omega) : \mathrm{supp}(f) \text{ is compact in } \Omega\}$,
- $C^\infty(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is infinitely differentiable}\}$,
- $\mathscr{D}(\Omega) = C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) : \mathrm{supp}(f) \text{ is compact in } \Omega\}$,

where

- $\mathrm{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$.
- Note that $\partial\Omega = \overline{\Omega} \setminus \Omega$
- A subset K of Ω is compact iff every open cover $\{\mathscr{O}_\alpha\}_{\alpha \in A}$ of K contains a finite subcover, that is, there exists a finite collection $\{\mathscr{O}_{\alpha_1}, \cdots, \mathscr{O}_{\alpha_m}\}$, $\alpha_1, \cdots, \alpha_m \in A$ such that $\bigcup_{i=1}^m \mathscr{O}_{\alpha_i} \supset K$. In case $\overline{U} \subset \Omega$ and $\overline{U}$ is compact, we denote $U \Subset \Omega$.

Notations

$$
\begin{aligned}
C^m(\overline{\Omega}) &= \{f : \Omega \to \mathbb{C} : \text{there exist an open set } \mathscr{O} \text{ containing } \Omega \text{ and } \tilde{f} \in C^m(\mathscr{O}) \text{ such that } f = \tilde{f}|_\Omega\}, \\
\mathscr{D}(\overline{\Omega}) &= \{f : \Omega \to \mathbb{C} : \text{there exist an open set } \mathscr{O} \text{ containing } \text{ and } \tilde{f} \in \mathscr{D}(\mathscr{O}) \text{ such that } f = \tilde{f}|_\Omega\}, \\
\mathbf{R}_+^d &= \{x = (x', x_d) \in \mathbf{R}^d : x_d > 0,\ x' \in \mathbf{R}^{d-1}\}, \\
\mathbf{R}_-^d &= \{x = (x', x_d) \in \mathbf{R}^d : x_d < 0,\ x' \in \mathbf{R}^{d-1}\}.
\end{aligned}
$$

Example 3.1. *Notice that the spaces of polynomials, trigonometric functions, and exponential functions are dense in $C^\infty(\Omega)$ under suitable norms.*

The following function

$$
\varphi(x) = \begin{cases} \exp\left(\dfrac{1}{|x-a|^2 - r^2}\right), & \text{if } |x-a| < r, \\ 0, & \text{otherwise.} \end{cases}
$$

belongs to $C_0^\infty(\mathbf{R}^d)$.

Definition 3.1. *A function* $\|\cdot\| : X \mapsto \mathbb{R}^+$ *is called a* *norm* *if*

$$\begin{array}{llll} (i) & \|x+y\| \le \|x\| + \|y\| & \forall x, y \in X & \\ (ii) & \|\alpha x\| = |\alpha|\,\|x\| & \forall x, \in X & \forall \alpha \in \mathbb{R} \\ (iii) & \|x\| > 0 & \forall x,\, x \neq 0. & \end{array}$$

Definition 3.2. *A function* $|\cdot| : X \mapsto \mathbb{R}^+$ *is called a* *semi-norm* *if (i) and (ii) hold in Definition 3.1.*

Theorem 3.1. *[Rud91a, p. 30] A topological vector space X is normable if and only if the origin has a convex bounded neighborhood.*

For an open subset Ω of $\mathbb{R}^d$,

$$\begin{aligned} L^p(\Omega) \;&:=\; \{f : \Omega \mapsto \mathbb{R} \mid \int_\Omega |f(x)|^p dx < \infty\} \\ &\quad \|f\|_{0,p,\Omega} := \left[\int_\Omega |f(x)|^p dx\right]^{\frac{1}{p}} \\ L^\infty(\Omega) \;&:=\; \{f : \Omega \mapsto \mathbb{R} \mid ess\sup_\Omega |f| < \infty\} \\ &\quad \|f\|_{0,\infty,\Omega} := ess\sup_\Omega |f| \end{aligned}$$

Then $L^p(\Omega)$ is Banach space for $p > 0$ for the norm $\|\cdot\|_{0,p,\Omega}$.
(*i.e.*, a vector space eqipped with a norm with which the space is complete.)

If $p = 2$, then $L^2(\Omega)$ is Hilbert space with the scalar product $(\cdot,\cdot)_{0,2,\Omega}$.
(*i.e.*, a vector space with an inner product (scalar product) which generates a norm with which the space is complete.)

$$(f,g)_{0,2,\Omega} \;:=\; \int_\Omega f(x)\, g(x)\, dx, \quad \|f\|_{0,2,\Omega} \;=\; \sqrt{(f,f)_{0,2,\Omega}}$$

Definition 3.3. *For Banach spaces* V, W*, denote by* $\mathscr{L}(V,W)$ *the set of all bounded linear mappings from* V *into* W*. If* $W = V$*, we use the abbreviation* $\mathscr{L}(V) = \mathscr{L}(V,V)$*.*

3.1 Duality

Definition 3.4. *Let* V *be a vector space with inner product* $(\cdot,\cdot)$ *and let* A *be a bounded linear operator on* V*.* A^* *is called an* *adjoint* *of* A*, if for all* $u, v \in V$ *,*

$$(Au, v) = (u, A^*v) \quad or \quad (u, Av) = (A^*u, v).$$

If $A = A^*$*, then* A *is called* *self-adjoint*.

Definition 3.5. *Let* V *be a normed linear space. The dual space* V' *is the set of all bounded linear functionals* $v' : V \to \mathbb{C}$*, which forms a complete normed linear space, (a Banach space) with the norm given by*

$$\|v'\|_{V'} \;:=\; \sup_{v\in V, v\neq 0} \frac{|v'(v)|}{\|v\|_V} \tag{3.1a}$$

$$\;=\; \sup_{v\in V, v\neq 0, \|v\|_V \le 1} |v'(v)| \tag{3.1b}$$

$$\;=\; \sup_{v\in V, \|v\|_V = 1} |v'(v)| \tag{3.1c}$$

The duality paring $\langle\cdot,\cdot\rangle_{V',V}$ *between* V' *and* V *will be denoted by*

$$\langle v', v\rangle_{V',V} := v'(v) \quad \forall v \in V\; \forall v' \in V'.$$

Definition 3.6. *[Yos95, p.120] A sequence (v_n) in a normed linear space V is said to converge weakly to $v_\infty \in V$ if $\lim_{n\to\infty} \langle v', v_n\rangle_{V',V}$ exists and it is equal to $\langle v', v_\infty\rangle_{V',V}$ for every $v' \in V'$.*

Theorem 3.2. *[Yos95, p.121] A sequence (v_n) in a normed linear space V is said to converge weakly to $v_\infty \in V$ iff*

1. $\sup_{n\ge1} \|v_n\|_V < \infty$ *and*
2. $\lim_{n\to\infty} \langle v', v_n\rangle_{V',V} = \langle v', v_\infty\rangle_{V',V}$ *for all $v' \in D'$ for any strongly dense subset D' of V'.*

Definition 3.7. *[Yos95, p.125] A sequence (v'_n) in the dual space of V' of a normed linear space V is said to converge weakly* to $v'_\infty \in V'$ if $\lim_{n\to\infty} \langle v'_n, v\rangle_{V',V}$ exists and it is equal to $\langle v'_\infty, v\rangle_{V',V}$ for every $v \in V$.*

Theorem 3.3. *[Yos95, p.121] A sequence (v'_n) in the dual space of V' of a normed linear space V converge weakly* to $v'_\infty \in V'$ if iff*

1. $\sup_{n\ge1} \|v'_n\|_{V'} < \infty$ *and*
2. $\lim_{n\to\infty} \langle v'_n, v\rangle_{V',V} = \langle v'_\infty, v\rangle_{V',V}$ *for all v on D for any strongly dense subset D of V.*

Theorem 3.4. *If V is a normed linear space, the closed unit ball B' of V' is weak*-compact. The weak* topology is the weakest topology that makes all linear functional $v^* \to \langle v^*, v\rangle_{V',V}$ continuous.*

Theorem 3.5. *If V is a normed linear space, V' is a complete normed linear space, (a Banach space) with the dual norm $\|\cdot\|_{V'}$*

Theorem 3.6. *Let V be a normed linear space. The map $V' \to \mathbb{C} : v' \mapsto \langle v', v\rangle_{V',V}$ is a bounded linear functional on V' with the norm $\|v\|_V$.*

Let V be a Banach space, and V'' the dual space of V', which consists of the set of bounded linear functional on V'. Thus, every $v \in V$ can be regarded as an element in $v'' \in V''$. Morever, due to the above theorem, the embedding $\mathscr{I} : V \to V''$ defined by $\mathscr{I}(v) = v''$ is an isometric isomorphism and $\mathscr{I}(V)$ is a closed subspace of V''. In case $\mathscr{I}(V) = V''$, V is called a *reflexive* Banach space.

Definition 3.8. *Let V, W be Banach spaces. Let $T : \mathscr{D}(T) \subset V \to W$ be an unbounded operator with the domain $\mathscr{D}(T)$ dense in V. An unbounded adjoint operator $T^* : \mathscr{D}(T^*) \subset W' \to V'$ is defined as follows: First, the domain $\mathscr{D}(T^*)$ is defined by follows:*

$$D(T^*) = \{w' \in W' : \text{ there exists a } c \text{ such that } |\langle w', Tv\rangle_{W',W}| \le c\|v\|_V \quad \forall v \in D(T)\}. \tag{3.2}$$

It is clear that $\mathscr{D}(T^*)$ is a subspace of W'. Since the map $v \to \langle w', Tv\rangle_{W',W} : \mathscr{D}(T) \to \mathscr{C}$ is bounded and $\mathscr{D}(T)$ is dense in V, by Hahn-Banach theorem there exists a unique linear map $f : V \to \mathbb{C}$ such that

$$|f(v)| \le c\|v\|_V \forall v \in V.$$

Denote such a map (depending on w') by

$$T^* w' = f,$$

Then the following is fulfilled:

$$\boxed{\langle w', Tv\rangle_{W',W} = \langle T^* w', v\rangle_{V',V} \quad \forall v \in \mathscr{D}(T), \forall w' \in \mathscr{D}(T^*).}$$

In particular if V and W are identical Hilbert space and T is a bounded linear map in V, then

$$(T^* x, y) = (x, Ty) \quad \text{for all } x, y \in V.$$

Theorem 3.7. *If V and W are normed linear spaces. Then for each $T \in \mathscr{L}(V, W)$,*

$$\|T^*\|_{\mathscr{L}(W',V')} = \|T\|_{\mathscr{L}(V,W)}.$$

3.2 Annihilators

Definition 3.9. *Suppose $M \subset V$ is a subspace of a Banach space V, and $N \subset V'$ is a subspace of V'. The annihilatos $M^\perp$ and $N^\perp$ are defined as follows:*

$$\begin{aligned} M^\perp &= \{v' \in V' : \langle v', v\rangle_{V',V} = 0 \quad \forall v \in M\}, \\ {}^\perp N &= \{v \in V : \langle v', v\rangle_{V',V} = 0 \quad \forall v' \in N\}. \end{aligned}$$

Theorem 3.8. *Suppose $M \subset V$ is a subspace of a Banach space V, and $N \subset V'$ is a subspace of V'. Then*

1. ${}^\perp(M^\perp) = \overline{M}^{\|\cdot\|_V}$. $({}^\perp N)^\perp = \overline{N}^{weak^* V'}$.

2. *In addidtion, if M is closed, then*
$$M' = V'/M^\perp; \qquad (V/M)' = M^\perp.$$

3. *For $T \in \mathscr{L}(V, W)$,*
$$\mathscr{N}(T^*) = \mathscr{R}(T)^\perp; \qquad \mathscr{N}(T) = {}^\perp\mathscr{R}(T^*).$$

4. *For $T \in \mathscr{L}(V, W)$, $\mathscr{N}(T^*)$ is weak*-closed in W'*

5. *For $T \in \mathscr{L}(V, W)$, $\mathscr{R}(T)$ is dense in W if and only if T^* is one-to-one.*

6. *For $T \in \mathscr{L}(V, W)$, $\mathscr{R}(T^*)$ is weak*-dense in V' if and only if T is one-to-one.*

3.3 Big Theorems

Theorem 3.9 (Bohnenblust-Sobczyk Extension Theorem). *Let V be a linear space with a seminorm $|\cdot|_V$. Let M be a linear subspace of V and f is a linear functional defined on M with $|f(v)| \le |v|_V$ for all $v \in M$. Then there exists a linearly extended fuctional $\widetilde{f}$ defined on V such that $|\widetilde{f}(v)| \le |v|_V$ for all $v \in V$.*

Theorem 3.10 (Hahn-Banach Extension Theorem in Normed Linear Spaces). *Let V be a normed linear space with norm $\|\cdot\|_V$. Let M be a linear subspace of V and f is a continuous linear functional defined on M with $|f(v)| \le \|v\|_V$ for all $v \in M$. Then there exists a continuous linearly extended fuctional $\widetilde{f}$ defined on V such that $\|\widetilde{f}\|_{\mathscr{V}'} = \|f\|_{\mathscr{V}'}$.*

Theorem 3.11 (Banach Open Mapping Theorem in Banach Spaces). *Let V and W be Banach spaces. Let $T \in \mathscr{L}(V, W)$ with $T(V) = W$. Then T maps every open set of V onto an open set of W.*

Theorem 3.12 (Closed graph theorem). *Let V, W be Banach spaces. Then if a map $T : V \to W$ is continuous, then the graph*
$$G = \{(v, Tv) : v \in V\}$$
is closed in $V \times W$. Conversely, if a map $T : V \to W$ is linear and the graph
$$G = \{(v, Tv) : v \in V\}$$
is closed in $V \times W$, then the map $T : V \to W$ is continuous.

3.4 Compact operators

Definition 3.10. *For Banach spaces V, W, a linear map $T : V \to W$ is said to be compact if $\overline{T(B(\mathbf{0};1))}$ is compact in W, where $B(\mathbf{0};1)$ denotes the open unit ball in V.*

Notice the following facts:

1. A compact operator $T : V \to W$ is evidently bounded.
2. $T : V \to W$ is compact if and only if every bounded sequence $\{v^n\}$ in V contains a subsequence $\{v^{n_k}\}$ such that $\{Tv^{n_k}\}$ converges to some w in W.

Definition 3.11. *For a Banach space V, suppose that $T \in \mathscr{L}(V)$. The spectrum $\sigma(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. Any $\lambda \in \sigma(T)$ should then satisfy at least one of the following statement:*

1. *$\mathscr{R}(T - \lambda I) \neq V$;*
2. *$\mathscr{N}(T - \lambda I) \neq \{\mathbf{0}\}$; in this case λ is called an eigenvalue of T with $\dim(\mathscr{N}(T - \lambda I)) \geq 1$, and any $v \in \mathscr{N}(T - \lambda I)$ is called an eigenvector of T associated with the eigenvalue λ.*

The following theorems are standard. For instance see Rudin [Rud91b].

Theorem 3.13. *Let V, W be Banach spaces.*

1. *If $T \in \mathscr{L}(V, W)$ and $\mathscr{R}(T)$ is a finite dimensional subspace of W, then T is compact.*
2. *If $T \in \mathscr{L}(V, W)$ is compact and $\mathscr{R}(T)$ is a closed subspace of W, then $\mathscr{R}(T)$ is a finite dimensional subspace of W.*
3. *If $T \in \mathscr{L}(V)$ is compact, then the null space $\mathscr{N}(T - \lambda I))$ is of finite dimension for all nonzero λ.*
4. *If $T \in \mathscr{L}(V)$ is compact in an infinite dimensional Banach space V, then $0 \in \sigma(T)$.*
5. *If $S, T \in \mathscr{L}(V)$ and T is compact, the composition $S \circ T$ is compact.*
6. *If $T \in \mathscr{L}(V, W)$, then T is compact if and only if T^* is compact. [Schauder's Theorem]*
7. *If $T \in \mathscr{L}(V)$ is compact, then $T - \lambda I$ has closed range for all nonzero λ.*
8. *Suppose that $T \in \mathscr{L}(V)$ is compact, and $r > 0$. Let E be a set of eigenvalues λ of T such that $|\lambda| > r$. Then E is a finite set, and $\mathscr{R}(T - \lambda I) \neq V$ for each $\lambda \in E$.*
9. *Suppose that $T \in \mathscr{L}(V)$ is compact. Then for all nonzero λ,*
$$\dim(\mathscr{N}(T - \lambda I)) = \dim(\mathscr{N}(T^* - \lambda I)) = \dim(V/\mathscr{R}(T - \lambda I)) = \dim(V'/\mathscr{R}(T^* - \lambda I)) < \infty.$$
10. *Suppose that $T \in \mathscr{L}(V)$ is compact. If $\lambda \in \sigma(T)$ is nonzero, then λ is an eigenvalue of T and T^*.*
11. *Suppose that $T \in \mathscr{L}(V)$ is compact. Then $\sigma(T)$ is compact, at most countable, and has possibly at most one limit point 0. Each nonzero eigenvalue has finite multiplicity.*

Theorem 3.14 (Fredholm Alternative)**.** *Let $T \in \mathscr{L}(V)$ be a compact operator in a normed linear space V. Then either*

1. *the homogeneous equation*
$$x - Tx = 0$$
has a nontrivial solution or

2. *for each* $y \in V$*, the nonhomogeneous equation*

$$x - Tx = y$$

is uniquely solvable. In this case, $(I - T)^{-1}$ *is also bounded.*

Proof. See [GT83]. □

Theorem 3.15 (Fredholm Alternative in Hilbert spaces)**.** *Let* $T \in \mathscr{L}(V)$ *be a compact operator in a (real) Hilbert space. Then there exists a countable set* $\Lambda \subset \mathbb{R}$ *with no limit point except possibly* $\lambda = 0$*, such that*

1. *for each nonzero* $\lambda \notin \Lambda$ *the equations*

$$\lambda x - Tx = y, \quad \lambda x - T^*x = y \tag{3.3}$$

hava a unique solution $x \in V$ *for each* $y \in V$*. Moreover* $(\lambda I - T)^{-1}$ *and* $(\lambda I - T^*)^{-1}$ *are bounded.*

2. *for each* $\lambda \in \Lambda$*,*

$$\lambda x - Tx = y \text{ is solvable if and only if } y \perp \mathscr{N}(\lambda I - T^*),$$

and

$$\lambda x - T^*x = y \text{ is solvable if and only if } y \perp \mathscr{N}(\lambda I - T);$$

moreover, $\mathscr{N}(\lambda I - T)$ *and* $\mathscr{N}(\lambda I - T^*)$ *are of finite dimension.*

Proof. See [GT83]. □

Definition 3.12. *For more general linear operator* $T : VtoV$ *with* $D(T) \subset V$*, set*

$$T_\lambda = \lambda I - T, \quad \lambda \in \mathbb{C}.$$

The resolvent set of T *is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} \mid R(T_\lambda) \text{ is dense in } V \text{ and } (T_\lambda)^{-1} \text{ exists and is continuous}\}$$

For $\lambda \in \rho(T)$*, the resolvent of* T *is defined by* $(T_\lambda)^{-1}$ *and denoted by* $R(\lambda : T)$*. The spectrum of* T *is the complement of* $\rho(T)$ *in* $\mathbb{C}$*, i.e.*

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

There are three kinds of spectra which are disjoint:

Point spectrum $P_\sigma(T) = \{\lambda \in \mathbb{C} \mid T_\lambda \text{ has no inverse}\} := \text{the set of eigenvalues of } T$*;*

Continuous spectrum $C_\sigma(T) = \{\lambda \in \mathbb{C} \mid T_\lambda \text{ has discontinuous inverse } T_\lambda^{-1}, D(T_\lambda^{-1}) \text{ is dense in } V\}$*;*

Residual spectrum $R_\sigma(T) = \{\lambda \in \mathbb{C} \mid T_\lambda \text{ has an inverse } T_\lambda^{-1}, D(T_\lambda^{-1}) \text{ is not dense in } V\}$*.*

Theorem 3.16. *[Yos95, p.209] If* $T : V \to V$ *is a closed linear operator, for any* $\lambda \in \rho(T)$ *the resolvent* $R(\lambda; T)$ *is an everywhere defined continuous linear operator, i.e.* $R(\lambda; T) \in \mathscr{L}(T)$*.*

Chapter 4

Sobolev Spaces

Multi-index and partial derivatives Let $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\alpha_i \geq 0$ integers, be a multi-index, and set $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Denote by $\partial^\alpha \varphi$ the partial derivative

$$D^\alpha \varphi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

4.1 Distributions

Definition of Distribution

Definition 4.1. *$(\phi_j)_j \subset C_0^\infty(\Omega)$ is said to converge in the sense of the space $\mathscr{D}(\Omega)$ to the function $\phi \in C_0^\infty(\Omega)$ if*

(i) *there exists a compact subset K of Ω such that* $\mathrm{supp}(\phi_j) \subset K \quad \forall j$;

(ii) *for all multi-index α, $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly on K,*
(i.e. $\forall\ \epsilon > 0$, $\exists$ an integer $N(\epsilon)$ such that $n \geq N(\epsilon) \longrightarrow |\partial^\alpha \phi_m(x) - \partial^\alpha \phi(x)| < \epsilon \quad \forall x \in K$.)

From now on, designate by $\mathscr{D}(\Omega)$ the space $C_0^\infty(\Omega)$ equipped with the topology structure (complete locally convex topological vector space) given by (i) and (ii).

Definition 4.2. *The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of distributions.*

The dual space $\mathscr{D}'(\Omega)$ is obviously the linear space (or vector space) of all continuous linear mapping $T : \mathscr{D}(\Omega) \to \mathbb{C}$. Thus if $S, T \in \mathscr{D}'(\Omega)$, $c \in \mathbb{C}$, then

$$\begin{aligned}(T+S)(\phi) &= T(\phi) + S(\phi),\\ (cT)(\phi) &= c(T(\phi)),\\ T(\phi + c\psi) &= T(\phi) + cT(\psi)\end{aligned}$$

for all $\phi, \psi \in \mathscr{D}(\Omega)$.

$T \in \mathscr{D}'(\Omega)$ if and only if

$$\lim_{j\to\infty} T(\phi_j) = T(\phi_j) \text{ whenever } \mathscr{D}(\Omega) - \lim_{j\to\infty} \phi_j = \phi \text{ in } \mathscr{D}(\Omega).$$

$T \in \mathscr{D}'(\Omega)$ if and only if for any compact subset $K \subset \Omega$ there exist $C_K > 0$ and $N_K > 0$ such that

$$T(\phi) \leq C_K \sup_K |\partial^\alpha \phi| \text{ for all } |\alpha| \leq N_K, \text{ whenever } \phi \in \mathscr{D}(\Omega) \text{ with } \mathrm{supp}(\phi) \subset K.$$

The topology structure on $\mathscr{D}'(\Omega)$ is given by the following convergence criterion: $T_j \to T$ in $\mathscr{D}'(\Omega) \Longleftrightarrow T_j(\phi) \to T(\phi)$ in $\mathbb{C}$ for all $\phi \in \mathscr{D}(\Omega)$. That is, the convergence of $T_j \to T$ in $\mathscr{D}'(\Omega)$ is weak* convergence.

Definition 4.3. *A function u defined almost everywhere on Ω is said to be* <u>*locally integrable*</u> *on Ω if $u \in L^1(K)$ for every measurable compact subset K of Ω. In this case, we denote by $u \in L^1_{\text{loc}}(\Omega)$.*

Example 4.1. $f(x) = |x|, \quad x \in (-\infty, \infty)$
f is not differentiable at 0, but f' is defined almost everywhere on $(-\infty, \infty)$ and $f' \in L^1_{loc}(-\infty, \infty)$.

Examples of distribution

Example 4.2. <u>Regular distributions :</u> *For all $u \in L^1_{\text{loc}}(\Omega)$, there is an associated distribution $T_u \in \mathscr{D}'(\Omega)$. Indeed, let $u \in L^1_{\text{loc}}(\Omega)$, and define the map $T_u : \mathscr{D}(\Omega) \to \mathbb{C}$ by*

$$T_u(\phi) = \int_\Omega \overline{u}\, \phi \, dx, \qquad \forall \phi \in \mathscr{D}(\Omega).$$

Then for $c \in \mathbb{C}$, $\phi, \psi \in \mathscr{D}(\Omega)$,

$$\begin{aligned} T_u(\phi + c\psi) &= \int_\Omega \overline{u}(\phi + c\psi) dx \\ &= \int_\Omega \overline{u}\phi dx + c \int_\Omega \overline{u}\psi dx \\ &= T_u(\phi) + cT_u(\psi). \end{aligned}$$

Therefore $T_u : \mathscr{D}(\Omega) \to \mathbb{C}$ is a linear map. To show that T_u is continuous, suppose that $\phi_j \to \phi$ in $\mathscr{D}(\Omega)$ as $j \to \infty$. Then there exists a $K \Subset \Omega$ such that $\text{supp}(\phi_j) \subset K$, and $\text{supp}(\phi) \subset K$. Thus,

$$|T_u(\phi_j) - T_u(\phi)| \le \sup_{x \in K} |\phi_j(x) - \phi(x)| \int_K |u(x)| dx \to 0$$

as $n \to \infty$, since $|\phi_j - \phi| \to 0$ uniformly. This shows that T_u is a distribution.

Example 4.3. <u>Dirac delta distribution.</u> *For $a \in \Omega$, define a linear form $\delta_a : \mathscr{D}(\Omega) \to \mathbf{R}$ by*

$$\delta_a(\phi) = \phi(a) \quad \forall \phi \in \mathscr{D}(\Omega).$$

Exercise 4.1. *Check that δ_a is a distribution.*

Example 4.4. <u>Heaviside function</u> *Define $H : \mathbf{R} \to \mathbf{R}$ by*

$$H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

Then $H \in L^1_{\text{loc}}(\mathbf{R})$. Therefore, by Example 4.2 there is an element $T_H \in \mathscr{D}'(\mathbf{R})$. Indeed it is given by

$$T_H(\phi) = \int_{\mathbf{R}} H(x)\phi(x) dx = \int_0^\infty \phi(x) dx.$$

Differentiation of distribution

Definition 4.4. *Given a distribution $T \in \mathscr{D}'(\Omega)$, its weak or distributional derivative (or derivative in the distribution sense) $\frac{\partial T}{\partial x_j} \in \mathscr{D}'(\Omega)$ is defined by*

$$\frac{\partial T}{\partial x_j}(\phi) = -T(\frac{\partial \phi}{\partial x_j}) \quad \forall \phi \in \mathscr{D}(\Omega).$$

In general, for $T \in \mathscr{D}'(\Omega)$, $\partial^\alpha T \in \mathscr{D}'(\Omega)$ is given by

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|}(\partial^\alpha \phi) \quad \forall \phi \in \mathscr{D}(\Omega). \tag{4.1}$$

The partial differentiation $\partial^\alpha : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ defined by (4.1) is continuous in the following sense:

$$\text{if } \; T_j \to T \text{ in } \mathscr{D}'(\Omega), \text{ then } \partial^\alpha T_j \to \partial^\alpha T \text{ in } \mathscr{D}'(\Omega).$$

Example 4.5. $u(x) = \frac{1}{2}|x|, \quad x \in (-\infty, \infty)$
(i) $T'_u : \mathscr{D}(\mathbf{R}) \mapsto \mathbf{R}$: *let* $\varphi \in \mathscr{D}(\mathbf{R})$ *be arbitrary. Then set* $K = \operatorname{supp}(\varphi)$, *which is compact.*

$$\begin{aligned} T'_u(\varphi) &= -\int_K \frac{1}{2}|x|\,\varphi'(x)\,dx = \frac{1}{2}\left[\int_{(-\infty,0)\cap K} x\,\varphi'(x)\,dx - \int_{(0,\infty)\cap K} x\,\varphi'(x)\,dx\right] \\ &= \frac{1}{2}\left[-\int_{(-\infty,0)\cap K} \varphi(x)\,dx + \int_{(0,\infty)\cap K} \varphi(x)\,dx\right] \\ &= \int_K (H(x) - \frac{1}{2})\,\varphi(x)\,dx. \end{aligned}$$

Consequently, $T'_u = T_{H-\frac{1}{2}}$ *in* $\mathscr{D}'(\mathbf{R})$
(ii) $T'_H : \mathscr{D}(\mathbf{R}) \mapsto \mathbf{R}$: $\quad \forall \varphi \in \mathscr{D}(\mathbf{R})$,

$$\begin{aligned} T'_H(\varphi) &= (-1)\,T_H(\varphi') = -\int_0^\infty \varphi'(x)\,dx \\ &= \varphi(0) = \delta_0(\varphi) \end{aligned}$$

where δ_0 *is Dirac delta distribution(function).*

Example 4.6. *If* $f : \mathbf{R} \to \mathbf{R}$ *has a continuous and bounded derivative in* $\mathbf{R} \setminus \{x_1, \cdots, x_m\}$, *with possible jumps at* $x_k, k = 1, \cdots, m$. *Let* $J_k = f(x_{k_+}) - f(x_{k_-})$, $k = 1, \cdots, m$. *Then*

$$\frac{d}{dx} T_f = T_{\widetilde{\frac{df}{dx}}} + \sum_{k=1}^m J_k \delta_{x_k},$$

where $\widetilde{\frac{df}{dx}}$ *is regarded as an* L^1_{loc} *function defined everywhere by the derivative of* f *except at* x_k*'s. Indeed, setting* $x_0 = -\infty, x_{m+1} = \infty$, *for all* $\phi \in \mathscr{D}(\mathbf{R})$,

$$\begin{aligned} \frac{d}{dx} T_f(\phi) = -\int_{\mathbf{R}} f(x) \frac{d\phi}{dx} dx &= \sum_{k=1}^{m+1} \int_{x_{k-1}}^{x_k} \frac{df}{dx}(x)\phi(x)dx + \sum_{k=1}^m J_k \phi(x_k) \\ &= \int_{\mathbf{R}} \widetilde{\frac{df}{dx}}(x)\phi(x)dx + \sum_{k=1}^m J_k \phi(x_k). \end{aligned}$$

Let $v \in L^2(\Omega)$. Then since $v \in L^1_{\text{loc}}(\Omega)$, we identify it with its corresponding distribution $T_v \in \mathscr{D}'(\Omega)$ defined in Example 4.2.

From now on, all the derivatives are understood in the distributional sense.

4.2 Sobolev Spaces

Definition 4.5. *For a nonnegative integer* m *and* $p \in [1, \infty]$ *the Sobolev norms are defined by*

$$\|u\|_{m,p,\Omega} := \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{p,\Omega}^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{4.2a}$$

$$\|u\|_{m,\infty,\Omega} := \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_{\infty,\Omega}. \tag{4.2b}$$

With the above Sobolev norms, there are two approaches to define Sobolev spaces of order (m, p) *as follows:*

$$H^{m,p}(\Omega) := \overline{\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}}^{\|\cdot\|_{m,p}}, \tag{4.3a}$$

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le m\}, \tag{4.3b}$$

$$W_0^{m,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)}, \tag{4.3c}$$

Theorem 4.1. *In general,*

$$H^{m,p}(\Omega) \subset W^{m,p}(\Omega).$$

The following salient theorem [MS64] removes pains to distinguish the above two definitions.

Theorem 4.2 (H=W, Meyers-Serrin (1964))**.** *For $p \in [1, \infty)$,*

$$H^{m,p}(\Omega) = W^{m,p}(\Omega).$$

Exercise 4.2. *Let $\Omega = (-1, 1)$ and $u(x) = |x|$. Then show that $u \in W^{1,\infty}(\Omega)$ but $u \notin H^{1,\infty}(\Omega)$.*

Due to Theorem 4.2, we will follow the definition of the original Sobolev spaces (due to Sobolev himself) $W^{m,p}(\Omega)$. In particular if $p = 2$, we will use the notation $H^m(\Omega)$ for $W^{m,2}(\Omega)$ so that

$$H^m(\Omega) = W^{m,2}(\Omega).$$

Theorem 4.3. *$W^{m,p}(\Omega)$ is a Banach space for $p \in [1, \infty]$ with the Sobolev norm $\|\cdot\|_{m,p,\Omega}$.*

Theorem 4.4. *$W^{m,p}(\Omega)$ is a separable Banach space for $p \in [1, \infty)$ and a reflexive Banach space for $p \in (1, \infty)$*

Corollary 4.1. *$H^m(\Omega)$ is a separable Hilbert space for all nonnegative integer m with the inner product*

$$(u, v)_{m,\Omega} = \sum_{0 \le |\alpha| \le m} \int_\Omega D^\alpha u D^\alpha \overline{v}\, dx$$

Recall that

Definition 4.6. *A topological vector space V is called separable if there exists a countable dense subset S of V.*

Thus Theorem 4.4 implies that there exists a countable subset $(f_n)_n$ of $W^{m,p}(\Omega)$ such that for any $f \in W^{m,p}(\Omega)$, there exists $(\alpha_n)_n \in \mathbb{C}$ such that

$$\left\| f - \sum_{n=1}^{\infty} \alpha_{nn} \right\|_{m,p,\Omega} = 0.$$

Denote by $|\cdot|_{m,p,\Omega}$ the Sobolev seminorm defined by

$$|f|_{m,p,\Omega} = \left[\sum_{|\alpha|=m} \|\partial^\alpha f\|_{0,p,\Omega} \right]^{\frac{1}{p}}.$$

Notice that

$$|f|_{m,p,\Omega} = 0 \qquad \text{if} \quad f(x) = \text{polynomial of order } m-1 \text{ in } \Omega.$$

The spaces $H^1(\Omega)$ and $W^{1,p}(\Omega)$

Example 4.7. *The Sobolev space of order $(1, 2)$ on Ω is given by*

$$H^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_j} \in L^2(\Omega), \quad j = 1, \cdots, d\},$$

equipped with the inner product

$$(u, v)_{1,\Omega} = \int_\Omega u\overline{v} dx + \sum_{k=1}^{d} \int_\Omega \frac{\partial u}{\partial x_k} \frac{\partial \overline{v}}{\partial x_k} dx, \quad u, v \in H^1(\Omega).$$

and the norm

$$||u||_{1,\Omega} = [(u, u)_{1,\Omega}]^{\frac{1}{2}}.$$

Example 4.8. *If $m = 1$,*

$$\|f\|_{1,p,\Omega} = \left[\|f\|^p_{0,p,\Omega} + \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x}\right\|^p_{0,p,\Omega}\right]^{\frac{1}{p}} = \left[\int_\Omega |f|^p + \sum_{j=1}^{n}\left|\frac{\partial f}{\partial x}\right|^p dx\right]^{\frac{1}{p}},$$

where the derivatives are understood in the sense of distribution.

Example 4.9. *$H^1(\Omega)$ is a Hilbert space for the inner product $(\cdot,\cdot)_{1,\Omega}$.*

Proof. Suppose $\{v_j\} \subset H^1(\Omega)$ is a Cauchy sequence. Then $\{v_j\}$ and $\{\frac{\partial v_j}{\partial x_k}\}$ for $k = 1,\cdots,d$ are Cauchy sequences in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there exist $v \in L^2(\Omega)$ and $v^{(k)} \in L^2(\Omega)$, $1 \le k \le d$, such that $v_j \to v$ in $L^2(\Omega)$ and $\frac{\partial v_j}{\partial x_k} \to v^{(k)}$ in $L^2(\Omega)$.

We have to prove that $v^{(k)} = \frac{\partial v}{\partial x_k}$ in the sense of distribution on Ω. Since $\mathscr{I} : L^2(\Omega) \to \mathscr{D}'(\Omega)$ defined by $\mathscr{I}(\phi) = \phi$ for all $\phi \in L^2(\Omega)$ is continuous,

$$\begin{aligned} \mathscr{I} v_j &\to \mathscr{I} v \quad &&\text{in} \quad \mathscr{D}'(\Omega) \\ \mathscr{I}\frac{\partial v_j}{\partial x_k} &\to \mathscr{I} v^{(k)} \quad &&\text{in} \quad \mathscr{D}'(\Omega)\ \ k = 1,\cdots,d. \end{aligned}$$

Since $\frac{\partial}{\partial x_k} : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is continuous,

$$\mathscr{I}\frac{\partial v_j}{\partial x_k} = \frac{\partial \mathscr{I} v_j}{\partial x_k} \to \frac{\partial \mathscr{I} v}{\partial x_k} = \mathscr{I}\frac{\partial v}{\partial x_k} \quad \text{in } \mathscr{D}'(\Omega).$$

Since the limit is unique in $\mathscr{D}'(\Omega)$, we have $\mathscr{I}\frac{\partial v}{\partial x_k} = \mathscr{I} v^{(k)} \in \mathscr{D}'(\Omega)$; and hence $\frac{\partial v}{\partial x_k} = v^{(k)} \in L^2(\Omega)$ for each $k = 1,\cdots,d$.

Include a commuting diagram here.

Therefore $v \in H^1(\Omega)$, and

$$||v_j - v||_{1,\Omega} = \left[\int_\Omega |v_j - v|^2 dx + \sum_{k=1}^{d}\int_\Omega |\frac{\partial v_j}{\partial x_k} - \frac{\partial v}{\partial x_k}|^2 dx\right]^{\frac{1}{2}} \to 0 \ \ \text{as} j \to \infty,$$

which implies that $v_j \to v$ in $H^1(\Omega)$ as $j \to \infty$. □

Example 4.10. *$H^1(\Omega)$ is separable (i.e. there exists a countable dense subset in $H^1(\Omega)$.)*

For a general domain Ω, $D(\Omega)$ is not dense in $H^1(\Omega)$. The closed set $\mathbf{R}^d \setminus \Omega$ should satisfy (m,p')-polar condition, for example; i.e., a distribution $T \in \mathscr{D}(\Omega)$ having support in $\mathbf{R}^d \setminus \Omega$ is the zero distribution. However, we recall Definition 4.5, from which we have

Definition 4.7. $H^1_0(\Omega) = \overline{\mathscr{D}(\Omega)}^{||\cdot||_{1,\Omega}}$

Corollary 4.2. *$\mathscr{D}(\mathbf{R}^d)$ is dense in $H^1(\mathbf{R}^d)$ i.e., $H^1_0(\mathbf{R}^d) = H^1(\mathbf{R}^d)$.*

Proof. □

Corollary 4.3. *For $1 \le p < \infty$, $W^{m,p}(\Omega)$ is separable,*

Theorem 4.5. *$\mathscr{D}(\Omega)$ is dense in $W^{m,p}(\Omega)$ if and only if the complement Ω^c is (m,p')-polar.*

4.2.1 Schwartz space $\mathfrak{S}(\mathbf{R}^d)$

$$\begin{aligned}\mathfrak{S}(\mathbf{R}^d) &:= \{f \in C^\infty(\mathbf{R}^d) \mid \sup_{\mathbf{x}\in\mathbf{R}^d} |\mathbf{x}^{\boldsymbol\beta}\partial^{\boldsymbol\alpha} f(\mathbf{x})| < \infty \quad \forall \boldsymbol\alpha, \boldsymbol\beta\} \\ &= \{f \in C^\infty(\mathbf{R}^d) \mid \sup_{\mathbf{x}\in\mathbf{R}^d} |(1+|\mathbf{x}|^2)^N \partial^{\boldsymbol\alpha} f(\mathbf{x})| < \infty \quad \forall \boldsymbol\alpha, N = 0,1,2,\cdots\}.\end{aligned}$$

which is the space of rapidly decreasing functions at ∞. Examples include any function in $C_0^\infty(\mathbf{R}^d)$, $e^{-|\mathbf{x}|^2}$, and so on.

For each polynomial $p(\mathbf{x})$ and multi-index $\boldsymbol\alpha$, $\mathscr{N} : \mathfrak{S}(\mathbf{R}^d) \to \mathbf{R}^+$ defined by

$$\mathscr{N}_p(f) := \sup_{\mathbf{x}\in\mathbf{R}^d} |p(\mathbf{x})\partial^{\boldsymbol\alpha} f(\mathbf{x})|$$

forms a seminorm on $\mathfrak{S}(\mathbf{R}^d)$. Indeed, due to the rapidly decreasing nature, the seminorms are norms on $\mathfrak{S}(\mathbf{R}^d)$.

Proposition 4.1. *[Yos95, p. 146] $\mathfrak{S}(\mathbf{R}^d)$ is closed with respect to the application of linear partial differential operators with polynomial coefficients.*

Proposition 4.2. *[Yos95, p. 146] $C_0^\infty(\mathbf{R}^d)$ is a dense subset of $\mathfrak{S}(\mathbf{R}^d)$ with respect to the topology of $\mathfrak{S}(\mathbf{R}^d)$.*

4.2.2 Fourier transformation

For any $f \in \mathfrak{S}(\mathbf{R}) \cup L^1(\mathbf{R})$, the Fourier transform $\widehat{f}$ is defined by $\widehat{f}(\omega) := \int_{-\infty}^\infty f(t)e^{-i\omega t}dt$, with the inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(\omega)e^{i\omega t}d\omega.$$

For higher dimension, Fourier transforms are defined analogously: for $f \in \mathfrak{S}(\mathbf{R}^d) \cup L^1(\mathbf{R}^d)$, the $\widehat{f} : \mathbf{R}^d \mapsto \mathbb{C}$ is defined by

$$\mathfrak{F}(f)(\boldsymbol\omega) = \widehat{f}(\boldsymbol\omega) := \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(\mathbf{x})e^{-i\boldsymbol\omega\cdot\mathbf{x}}dx_1\cdots dx_d$$

with the inversion formula

$$\mathfrak{F}^{-1}(\widehat{f})(\mathbf{x}) := \left(\frac{1}{2\pi}\right)^d \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \widehat{f}(\boldsymbol\omega)e^{i\boldsymbol\omega\cdot\mathbf{x}}d\omega_1\cdots d\omega_d. \tag{4.4}$$

Here, $\boldsymbol\omega = (\omega_1, \cdots, \omega_d)$ and $\mathbf{x} = (x_1, \cdots, x_d)$.

Proposition 4.3. *[Yos95, p. 146] In particular, both Fourier transformation and its inverse transformation are linear and continuous from $\mathfrak{S}(\mathbf{R}^d)$ to $\mathfrak{S}(\mathbf{R}^d)$. That is, the Fourier transform $\mathfrak{F} : \mathfrak{S} \to \mathfrak{S}$ which maps ϕ to $\widehat{\phi}$ is an isomorphism.*

Theorem 4.6 (Fourier Integral Theorem)**.** *[Yos95, p. 147]*

$$\mathfrak{F}^{-1} \circ \mathfrak{F} = I_{\mathfrak{S}(\mathbf{R}^d)}.$$

One of merits of Fourier transformation is to transfer derivatives in $f(\mathbf{x})$ into algebraic factor in $\widehat{f}(\boldsymbol\omega)$. Given $f \in W^{m,2}(\Omega)$

$$\begin{aligned}\sum_{j=1}^d \int_\Omega \left|\frac{\partial f}{\partial x_j}\right|^2 d\mathbf{x} < \infty &\iff \sum_{j=1}^d \int_{\mathbf{R}} \left|\widehat{\frac{\partial f}{\partial x_j}}\right|^2 d\boldsymbol\omega < \infty \\ &\iff \sum_{j=1}^d \int_{\mathbf{R}} \omega_j^2 \left|\widehat{f}(\boldsymbol\omega)\right|^2 d\boldsymbol\omega < \infty \\ &\iff \int_{\mathbf{R}} |\boldsymbol\omega|^2 \left|\widehat{f}(\boldsymbol\omega)\right|^2 d\boldsymbol\omega < \infty\end{aligned}$$

Theorem 4.7 (Parseval formula)**.**

$$\|\widehat{f}\|_0^2 = (2\pi)^d \|f\|_0^2. \tag{4.5}$$

Theorem 4.8 (Plancherel Theorem)**.** *The Fourier transform, defined originally on $L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ extends uniquely to a map from $L^2(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$ satisfying*

$$(\widehat{f}, \widehat{g}) = (2\pi)^d (f, g), \quad \|\widehat{f}\|_0^2 = (2\pi)^d \|f\|_0^2, \tag{4.6}$$

for all $f, g \in L^2(\mathbf{R}^d)$.

We will extend the Fourier transformation to (*temperate*) distributions.

The space $\mathfrak{S}$ is endowed with the topology defined by the seminorm

$$|\phi|_{\mathfrak{S}} := \sup_{\alpha,\beta,x} |x^\beta \partial^\alpha \phi(x)|$$

makes $\mathfrak{S}$ a Fréchet space.

A space V is called a Fréchet space if it is a locally convex space whose topology is induced by a complete invariant metric $\operatorname{dist}(\cdot, \cdot)$.

Definition 4.8. *A temperate distribution $\mathfrak{S}'$ is the set of all continuous linear functional u on $\mathfrak{S}$.*

We are now in a position to define the Fourier transformation for temperate distributions.

Definition 4.9. *For $u \in \mathfrak{S}'$, the Fourier transform $\widehat{u}$ is defined by*

$$\widehat{u}(\phi) = u(\widehat{\phi}) \quad \forall \phi \in \mathfrak{S}. \tag{4.7}$$

We then have the following theorems.

Theorem 4.9. *The Fourier transformation $\mathfrak{F} : \mathfrak{S}' \to \mathfrak{S}'$ is an isomorphism (with the weak topology) with the Fourier inversion of the form* (4.4) *for all $u \in \mathfrak{S}'$.*

Example 4.11. *[Rud91a, p.190]*

1. *All distributions with compact support are temperate distributions.*
2. *$L^p(\mathbf{R}^s) \subset \mathfrak{S}(\mathbf{R}^d)$ for $p \in [1, \infty]$.*
3. *$P_k(\mathbf{R}^s) \subset \mathfrak{S}(\mathbf{R}^d)$, $k = 0, 1, \cdots$. The polynomials are temperate distributions.*
4. *A Borel measure μ on $\mathbf{R}^d$ such that*

$$\int_{\mathbf{R}^d} \frac{1}{(1 + |\mathbf{x}|^2)^k} d\mu(\mathbf{x}) < \infty,$$

for some positive integer k, is a temperate distribution.

5. *A measurable function g on $\mathbf{R}^d$ such that*

$$\int_{\mathbf{R}^d} \frac{|g(x)|^p}{(1 + |\mathbf{x}|^2)^k} d\mathbf{x} < \infty,$$

for some positive real number k, is a temperate distribution.

Theorem 4.10. *Let $u \in \mathfrak{S}'$ and $v \in D'(\mathbf{R}^d)$ with $\operatorname{supp}(v)$ compact. Then $u * v \in \mathfrak{S}'$ and the Fourier transform $\mathfrak{F}(u * v)$ is equal to $\widehat{u}\widehat{v}$.*

Property 4.1. *For $u \in \mathfrak{S}'$, the following holds:*

1. $\widehat{\frac{d}{dx_j}u} = i\omega_j \widehat{u}$.

2. $\widehat{x_j u} = -i\frac{d}{d\omega_j}\widehat{u}$.

For any real number s, the Sobolev space can be defined through Fourier transformation:

$$H^s(\mathbf{R}^d) := \{f : \mathbf{R}^d \mapsto \mathbf{R} \mid \int_{\mathbf{R}^d} (1+|\boldsymbol{\omega}|^2)^s |\widehat{f}(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} < \infty\}$$

Example 4.12. *Every distribution with compact support is in* $H^s(\mathbf{R}^d)$.

Definition 4.10. *Let* $X = \cup_{s\in\mathbf{R}} H^s(\mathbf{R}^d)$. *A linear operator* $T : X \to X$ *is said to have order* t *if* $T_{H^s(\mathbf{R}^d)} : H^s(\mathbf{R}^d) \to H^{s-t}(\mathbf{R}^d)$ *is a continuous surjective mapping.*

Example 4.13. 1. *Every partial differential operator* $\partial^{\boldsymbol{\alpha}}$ *is an operator with order* $|\boldsymbol{\alpha}|$.

2. *For* $t \in \mathbf{R}$, *the mapping* $T : u \mapsto v$ *given by*

$$\widehat{v}(\boldsymbol{\omega}) = (1+|\boldsymbol{\omega}|^2)^{\frac{t}{2}} \widehat{u}(\boldsymbol{\omega})$$

is a linear isometry of $H^s(\mathbf{R}^d)$ *onto* $H^{s-t}(\mathbf{R}^d)$ *with order* t. *The inverse mapping* $T^{-1} : u \mapsto v$ *given by*

$$\widehat{v}(\boldsymbol{\omega}) = (1+|\boldsymbol{\omega}|^2)^{-\frac{t}{2}} \widehat{u}(\boldsymbol{\omega})$$

is a linear isometry of $H^s(\mathbf{R}^d)$ *onto* $H^{s+t}(\mathbf{R}^d)$ *with order* $-t$.

4.2.3 Prolongation (Extension)

Theorem 4.11. *If* $v \in H_0^1(\Omega)$, *the function*

$$\widetilde{v}(x) = \begin{cases} v(x) & x \in \Omega \\ 0 & x \in \mathbf{R}^d \setminus \Omega \end{cases}$$

belongs to $H^1(\mathbf{R}^d)$.

Proof. If $\varphi \in \mathscr{D}(\Omega)$, then

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & x \in \Omega \\ 0 & x \in \mathbf{R}^d \setminus \Omega \end{cases}$$

belongs to $\mathscr{D}(\mathbf{R}^d)$. Moreover,

$$\|\widetilde{\varphi}\|_{1,\mathbf{R}^d} = \|\varphi\|_{1,\Omega}.$$

The map $\mathscr{D}(\Omega) \to H^1(\mathbf{R}^d) : \varphi \mapsto \widetilde{\varphi}$ is linear and continuous with respect to the norm induced by $\|\cdot\|_{1,\Omega}$. Therefore this map extends to the map $H_0^1(\Omega) \to H^1(\mathbf{R}^d) : v \mapsto \widetilde{v}$ continuously and linearly. □

Lemma 4.1 (Poincaré inequality)**.** *If* Ω *is bounded, then there exists a constant* $C = C(\Omega)$ *such that*

$$||v||_{0,\Omega} \le C \left[\sum_{j=1}^d \left\| \frac{\partial v}{\partial x_j} \right\|_{0,\Omega}^2 \right]^{1/2} \qquad \forall\ v \in H_0^1(\Omega).$$

Remark 4.1. $v = 1$ *on* Ω *does not satisfy Poincaré inequality. Thus* $v \in H^1(\Omega)$ *is not sufficient.*

Corollary 4.4 (Poincaré Inequality).

$$\|v\|_{1,2,\Omega} \le C \left[\sum_{j=1}^{d} \left\| \frac{\partial v}{\partial x_j} \right\|_{0,2,\Omega}^2 \right]^{\frac{1}{2}} \qquad \forall v \in H_0^1(\Omega), \tag{4.8}$$

where C is independent of v, but depends only on Ω.

Remark 4.2. *This corollary is a general version for Lemma 4.1 since*

$$\|v\|_{1,\Omega}^2 = \|v\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2 \le C^2 \|\nabla v\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2 = (C^2+1)\|\nabla v\|_{0,\Omega}^2 .$$

In $H_0^1(\Omega)$, by Corollary 4.5

$$\|\cdot\|_{H^1(\Omega)} \simeq |\cdot|_{H^1(\Omega)}, \ \textit{i.e.}, \ \|v\|_{H^1(\Omega)} \le C_1 |v|_{H^1(\Omega)} \le C_2 \|v\|_{H^1(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

Corollary 4.5. *If Ω is bounded, the seminorm*

$$|v|_{1,\Omega} := \left[\sum_{j=1}^{d} \| \frac{\partial v}{\partial x_j} \|_{0,\Omega}^2 \right]^{1/2}$$

is equivalent to the Sobolev norm $||\cdot||_{1,\Omega}$ in $H_0^1(\Omega)$.
(i.e. $C\|v\|_1 \le |v|_1 \le C'\|v\|_1$ for some $C, C' \in \mathbf{R}$)

4.2.4 The definition of traces and trace theorems

For 1-dimensional case, let $\Omega = (a,b)$. Then if $v \in H^1(\Omega)$ then $v \in C^0(\overline{\Omega})$. In this case, it is not difficult to define $v|_\Gamma$ i.e., the values $v(a)$ and $v(b)$ for all $v \in H^1(\Omega)$.

However, for d-dimensional case, $d \ge 2$, it is not trivial to define boundary values of $v \in H^1(\Omega)$. Indeed, for $R < 1$, let $\Omega = \{x \in \mathbf{R}^2 : 0 < |x| < R\}$ and $v(x) = |\log|x||^p$. Then

$$\frac{\partial v}{\partial x_j} = p \left(\log \sqrt{x_1^2 + x_2^2} \right)^{p-1} \frac{x_j}{x_1^2 + x_2^2} \qquad \text{and}$$

$$\begin{aligned} ||v||_{1,\Omega}^2 &= \int |v|^2 + |\nabla v|^2 dx \\ &= 2\pi \int_0^R |\log r|^{2p} r dr + 2p^2 \pi \int_0^R |\log r|^{2p-2} \frac{r^2}{r^4} r\, dr \\ &< \infty \quad \text{if } p < \frac{1}{2} \end{aligned}$$

In order for $v \in H^1(\Omega)$ it suffices to choose $0 < p < \frac{1}{2}$; however, the function v can not be represented by a continuous function in $\overline{\Omega}$, since v is not bounded near the origin if $p > 0$.

Let $\Omega = \mathbf{R}_+^d$. Then $\Gamma = \{x = (x', 0); x' \in \mathbf{R}^{d-1}\}$.

Lemma 4.2. *$\mathscr{D}(\overline{\mathbf{R}_+^d})$ is dense in $H^1(\mathbf{R}_+^d)$.*

Proof. □

Definition 4.11. *For an integer $m \ge 1$, an open set $\Omega \subset \mathbf{R}^d$ is said to be m-regular if its boundary Γ is a manifold of class C^m of dimension $d-1$, Ω being locally at one side of Γ. Or equivalently, if there exists a finite open cover $\{\mathscr{O}_j\}_{j=1}^J$ of Ω and invertible $\varphi_j = \mathscr{O}_j \to B(0;1)$ such that φ_j and φ_j^{-1} are C^m maps and*

$$\begin{aligned} \varphi_j(\mathscr{O}_i \cap \Omega) &= B(0;1) \cap \mathbf{R}_+^d = \{y = (y', y_d) \in \mathbf{R}^d : |y'| < 1, y_d > 0\} \\ \varphi_j(\mathscr{O}_i \cap \Gamma) &= \{y = (y', y_d) \in \mathbf{R}^d : |y'| < 1, y_d = 0\}. \end{aligned}$$

Lemma 4.3. *If Ω is 1-regular, then there exists a continuous linear extension operator $\mathscr{E} : H^1(\Omega) \to H^1(\mathbf{R}^d)$ such that*

$$\mathscr{E}v(x) = v(x) \quad \forall x \in \Omega$$

Proof. (1st step) We begin by considering the case $\Omega = \mathbf{R}^d_+$. If $v \in \mathscr{D}(\overline{\mathbf{R}^d_+})$, define $\mathscr{E}v$ on $\mathbf{R}^d$ by (reflection)

$$\mathscr{E}v(x', x_d) = \begin{cases} v(x', x_d), & \text{if } x_d \ge 0 \\ v(x', -x_d), & \text{if } x_d \le 0. \end{cases}$$

Then $\mathscr{E}v$ is continuous, $\mathscr{E}v \in H^1(\mathbf{R}^d)$ and

$$\frac{\partial}{\partial x_j}\mathscr{E}v(x', x_d) = \begin{cases} \frac{\partial v}{\partial x_j}(x', x_d), & j = 1, \cdots, d-1, \\ \frac{\partial v}{\partial x_j}(x', x_d), & \text{if } x_d > 0,\ j = d, \\ -\frac{\partial v}{\partial x_j}(x', -x_d), & \text{if } x_d < 0,\ j = d. \end{cases}$$

Then for all $v \in \mathscr{D}(\overline{\mathbf{R}^d_+})$, we have

$$\|\mathscr{E}v\|_{1,\mathbf{R}^d} = \sqrt{2}\, \|v\|_{1,\mathbf{R}^d_+}\, .$$

Since $\mathscr{D}(\overline{\mathbf{R}^d_+})$ is dense in $H^1(\mathbf{R}^d_+)$, $\mathscr{E}$ can be extended by linearly and continuously to $\mathscr{E} : H^1(\mathbf{R}^d_+) \to H^1(\mathbf{R}^d)$.

(2nd step) Now assume that Ω is an open 1-regular domain in $\mathbf{R}^d$. We know that there exists a partition of unity $\{\alpha_j\}_{j=0}^J$ subordinate to the cover $\{\mathscr{O}_j\}_{j=1}^J$ of the boundary Γ of Ω; i.e.,

$$\alpha_j \in \mathscr{D}(\mathscr{O}_j),\ 1 \le j \le J; \quad \sum_{j=0}^J \alpha_j = 1 \text{ on } \overline{\Omega},\ 0 \le \alpha_j \le 1, \quad \text{and } \Gamma \subset \cup_{j=1}^J \mathscr{O}_j.$$

Thus, for $v \in H^1(\Omega)$ we can write

$$v = \sum_{j=0}^J \alpha_j v,$$

we can define $\mathscr{E}(\alpha_j v)$ for all $j = 0, 1, \cdots, J$, and then define $\mathscr{E}v$ by

$$\mathscr{E}v = \sum_{j=0}^J \mathscr{E}(\alpha_j v).$$

First, we have

$$\mathscr{E}(\alpha_0 v) = \widetilde{\alpha_0 v} : \text{ the extension of } \alpha_0 v \text{ by 0 to } \mathbf{R}^d \setminus \Omega.$$

Then for $j = 1, \cdots, J$, consider

$$w_j = (\alpha_j v) \circ (\varphi_j^{-1}|_{B_+}), \quad B_+ = B(0;1) \cap \mathbf{R}^d_+.$$

We have $w_j \in H^1(B_+)$ and $w_j = 0$ in a neighborhood of $\{y \in \partial B_+ : y_d > 0\}$.

One can extend w_j by 0 in $\mathbf{R}^d_+ \setminus B_+$. Let $\widetilde{w_j}$ denote this extension. Then $\widetilde{w_j} \in H^1(\mathbf{R}^d_+)$, which is again extended to $\mathbf{R}^d$ by reflection (as in the first step)

$$\widetilde{\widetilde{w_j}} \in H^1(\mathbf{R}^d) \quad \text{and } \operatorname{supp}(\widetilde{\widetilde{w_j}}) \subset B(0;1).$$

Then let $\widetilde{\widetilde{\widetilde{w_j}} \circ \varphi} \in H^1(\mathbf{R}^d)$: the extension of $\widetilde{\widetilde{w_j}} \circ \varphi$ by 0 in $\mathbf{R}^d \setminus \mathscr{O}_j$. We have

$$\mathscr{E}(\alpha_j v) = \widetilde{\widetilde{\widetilde{w_j}} \circ \varphi_j}, \quad 1 \le j \le J.$$

Then $v \mapsto \sum_{j=0}^J \mathscr{E}(\alpha_j v)$ is a 1-extension. □

Lemma 4.4. *If Ω is 1-regular, $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$.*

Proof. Let $v \in H^1(\Omega)$. Then by the Lemma 4.3, there exists a function $\mathscr{E}v \in H^1(\mathbf{R}^d)$ such that $\mathscr{E}v = v$ on Ω. By Theorem 4.10, $H^1_0(\mathbf{R}^d) = H^1(\mathbf{R}^d)$, there exists a sequence $\{w_j\}_j \subset \mathscr{D}(\mathbf{R}^d)$ such that $w_j \to \mathscr{E}v$ in $H^1(\mathbf{R}^d)$.

Let v_j be the restriction of w_j to Ω. Then

$$(v_j)_j \subset \mathscr{D}(\overline{\Omega}) \quad \text{s.t.} \quad v_j \to v \text{ in } H^1(\Omega).$$

Therefore $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$. □

Define

$$\|v\|_{0,\Gamma} = \left[\int_\Gamma |v(x)|^2 d\sigma\right]^{1/2}. \tag{4.9}$$

By using a partition of unity, $\{\alpha_k\}_{k=0}^K$, we have

$$L^2(\Gamma) = \{v : \Gamma \to \mathbf{R} : \widetilde{(\alpha_k v) \circ \varphi_k^{-1}}(\cdot, 0) \in L^2(\mathbf{R}^{d-1}), 1 \le k \le K\},$$

where $\widetilde{(\alpha_k v) \circ \varphi_k^{-1}}(\cdot, 0)$ is the extension by 0 to $\mathbf{R}^{d-1} \setminus \{y' \in \mathbf{R}^{d-1} : |y'| < 1\}$. Then, the map given by

$$v \mapsto \left[\sum_{k=0}^K \left\|\widetilde{(\alpha_k v) \circ \varphi_k^{-1}}\right\|^2_{0,\mathbf{R}^{d-1}}\right]^{1/2} \tag{4.10}$$

defines a norm which is equivalent to the norm given by $\|\cdot\|_{0,\Gamma}$ defined in (4.9).

Consider the mapping $\gamma_0 : \mathscr{D}(\overline{\Omega}) \to L^2(\Gamma)$ defined by

$$\gamma_0 v = v|_G \quad \forall v \in \mathscr{D}(\overline{\Omega}).$$

Lemma 4.5. *If Ω is 1-regular, then there exists $C > 0$ such that*

$$|\gamma_0 v|_{0,\Gamma} \le C||v||_{1,\Omega} \quad \forall\ v \in \mathscr{D}(\overline{\Omega}).$$

where C is independent of v, but dependent only on Ω.

Proof. Let $v \in \mathscr{D}(\overline{\Omega})$. By the partition of unity $\{\alpha_k\}_{k=0}^K$, we put

$$w_k = (\alpha_k v) \circ \varphi_k^{-1},\ 0 \le k \le K.$$

Since $||v(\cdot, 0)||_{0,\mathbf{R}^{d-1}} \le ||v||_{1,\mathbf{R}^d_+}$ for all $v \in \mathscr{D}(\mathbf{R}^d_+)$,

$$\|\widetilde{w_k}(\cdot, 0)\|_{0,\mathbf{R}^{d-1}} \le \|\widetilde{w_k}\|_{1,\mathbf{R}^d_+}.$$

Since α_k and φ_k^{-1} are smooth, we can find

$$\|\widetilde{w_k}\|_{1,\mathbf{R}^d_+} \le C_k \|v\|_{1,\Omega}.$$

Thus

$$||\widetilde{w_k}(\cdot, 0)||_{0,\mathbf{R}^{d-1}} \le C_k \|v\|_{1,\Omega}.$$

By the equivalence of the two norms $\|\cdot\|_{0,\Gamma}$ given by (4.9) and (4.10), we are done. □

Theorem 4.12 (Trace Theorem)**.** *Suppose that Ω is 1-regular. Then $\mathscr{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$ and $\gamma_0 : v \mapsto \gamma_0 v = v|_\Gamma$ from $\mathscr{D}(\overline{\Omega})$ to $L^2(\Gamma)$ can be extended linearly and continuously to a map, again denoted by γ_0*

$$\gamma_0 : H^1(\Omega) \to L^2(\Gamma).$$

Remark 4.3. *The trace theorem holds for domain $\Omega \subset \mathbf{R}^d$ which is Lipschitz domain or piecewise 1-regular.*

4.2.5 Application of Trace Theorem

Theorem 4.13. *Suppose Ω is open bounded in $\mathbf{R}^d$ with its boundary Γ being piecewise C^1. Then*

$$\begin{aligned} H_0^1(\Omega) &= Ker(\gamma_0) \quad i.e., \\ H_0^1(\Omega) &= \{v \in H^1(\Omega) : \gamma_0 v = v|_\Gamma = 0 \text{ on } \Gamma\}. \end{aligned}$$

Proof. (1) $H_0^1(\Omega) \subset \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. Indeed, let $v \in H_0^1(\Omega)$. Then there exists $\{\varphi_j\} \subset \mathscr{D}(\Omega)$ such that $\varphi_j \to v$ in $H^1(\Omega)$. Since γ_0 is continuous,

$$L^2(\Gamma) - \lim_{j\to\infty} \gamma_0\varphi_j = \gamma_0(H^1(\Omega) - \lim_{j\to\infty} v_j) = \gamma_0 v$$

Since $\gamma_0\varphi_j = 0$ on Γ for all j, we have $\gamma_0 v = 0$.

(2) To show $H_0^1(\Omega) \supset \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. Using the local property and partition of unity, it suffices to prove the case where $v \in H^1(\mathbf{R}_+^d)$ and has a compact support in $\overline{\mathbf{R}_+^d}$ satisfying $v(\cdot, 0) = 0$. It suffices to show that $v \in H_0^1(\mathbf{R}_+^d)$. Let $\widetilde{v}$ be the extension of v by 0 to $\mathbf{R}^d \setminus \mathbf{R}_+^d$. One can verify that $\widetilde{v} \in H^1(\mathbf{R}^d)$ by showing that

$$\frac{\partial \widetilde{v}}{\partial x_j} = \widetilde{\frac{\partial v}{\partial x_j}}, \quad 1 \le j \le d, \tag{4.11}$$

in the sense of distribution (we already know that $\frac{\partial v}{\partial x_j} \in L^2(\mathbf{R}_+^d)$ and hence $\widetilde{\frac{\partial v}{\partial x_j}} \in L^2(\mathbf{R}^d)$. Thus (4.11) implies $\frac{\partial \widetilde{v}}{\partial x_j} \in L^2(\mathbf{R}^d)$). In order to show (4.11), we need

Lemma 4.6. *If $v, \varphi \in H^1(\mathbf{R}_+^d)$, we have the following version of Green's formula*

$$\begin{aligned} \int_{\mathbf{R}_+^d} v\frac{\partial \varphi}{\partial x_j}dx &= -\int_{\mathbf{R}_+^d} \frac{\partial v}{\partial x_j}\varphi dx, \quad 1 \le j \le d-1, \\ \int_{\mathbf{R}_+^d} v\frac{\partial \varphi}{\partial x_d}dx &= -\int_{\mathbf{R}_+^d} \frac{\partial v}{\partial x_d}\varphi dx - \int_{\mathbf{R}^{d-1}} v(x',0)\varphi(x',0)dx'. \end{aligned}$$

We begin by obvserving that for $\varphi \in \mathscr{D}(\mathbf{R}^d)$,

$$\frac{\partial \widetilde{v}}{\partial x_j}(\varphi) = -\widetilde{v}(\frac{\partial \varphi}{\partial x_j}) = -\int_{\mathbf{R}^d} \widetilde{v}\frac{\partial \varphi}{\partial x_j}\, dx = -\int_{\mathbf{R}_+^d} v\frac{\partial \varphi}{\partial x_j}\, dx, \quad 1 \le j \le d.$$

Since $v(\cdot, 0) = 0$, Lemma 4.6 implies that

$$\frac{\partial \widetilde{v}}{\partial x_j}(\varphi) = \int_{\mathbf{R}_+^d} \frac{\partial v}{\partial x_j}\varphi\, dx = \widetilde{\frac{\partial v}{\partial x_j}}(\varphi), \quad 1 \le j \le d.$$

This proves (4.11), and hence $\widetilde{v} \in H^1(\mathbf{R}^d)$.

For all $h > 0$, consider the translattion operator $\tau_h : H^1(\mathbf{R}^d) \to H^1(\mathbf{R}^d)$ defined by

$$\tau_h\widetilde{v}(x', x_d) = \widetilde{v}(x', x_d + h).$$

Since $\frac{\partial}{\partial x_j}\tau_h\widetilde{v} = \tau_h\frac{\partial \widetilde{v}}{\partial x_j}$, $1 \le j \le d$, $\tau_h\widetilde{v} \in H^1(\mathbf{R}^d)$. Moreover, one can verify that $\tau_h\widetilde{v} \to \widetilde{v}$ in $H^1(\mathbf{R}^d)$ as $h \to 0$.

We introduce a regularization $\{\rho_\varepsilon\} \subset \mathscr{D}(\mathbf{R}^d)$. Then $\rho_\varepsilon * \tau_h\widetilde{v} \to \tau_h\widetilde{v}$ in $H^1(\mathbf{R}^d)$ as $\varepsilon \to 0$. And for sufficiently small $\varepsilon > 0$, $\rho_\varepsilon * \tau_h\widetilde{v}$ has a compact support in $\mathbf{R}_+^d$, from which one can construct a sequence of functions in $\mathscr{D}(\mathbf{R}_+^d)$ that converges to v in $H^1(\mathbf{R}_+^d)$. Therefore, $v \in H_0^1(\mathbf{R}_+^d)$.

Proof of Lemma 4.6. It is easy to see that (i) and (ii) hold for $v, \varphi \in \mathscr{D}(\overline{\mathbf{R}_+^d})$. Since $\mathscr{D}(\overline{\mathbf{R}_+^d})$ is dense in $H^1(\mathbf{R}_+^d)$ and the trace is continuous ($\gamma_0 : H^1(\mathbf{R}_+^d) \to L^2(\mathbf{R}^{d-1})$. □

Let $\nu = (\nu_1, \cdots, \nu_d)$ denote the unit outward normal to Γ.

Theorem 4.14. *Let Ω be open bounded with a piecewise C^1 boundary Γ. Then the map $v \mapsto (\gamma_0 v, \gamma_1 v) = (v|_\Gamma, \frac{\partial v}{\partial \nu}|_\Gamma)$ from $\mathscr{D}(\overline{\Omega})$ to $L^2(\Gamma) \times L^2(\Gamma)$ can be extended to a continuous linear map*

$$H^2(\Omega) \to L^2(\Gamma) \times L^2(\Gamma).$$

Theorem 4.15. *Let Ω be bounded open with piecewise C^1 boundary Γ. Then if $u, v \in H^1(\Omega)$,*

$$\int_\Omega \frac{\partial u}{\partial x_j} v dx = -\int_\Omega u \frac{\partial v}{\partial x_j} dx + \int_\Gamma \nu_j \gamma_0(uv) d\sigma, \quad 1 \le j \le d.$$

Proof. If $u, v \in H^1(\Omega)$, there exist $u_j, v_k \in \mathscr{D}(\overline{\Omega})$ such that

$$u_j \to u \text{ and } v_k \to v, \text{ in } H^1(\Omega)$$

For all $u_j, v_k \in \mathscr{D}(\overline{\Omega})$, we have

$$\int_\Omega \frac{\partial u_j}{\partial x_l} v_k dx = -\int_\Omega u_j \frac{\partial v_k}{\partial x_l} dx + \int_\Gamma \nu_l \gamma_0(u_j v_k) d\sigma, \quad 1 \le l \le d.$$

By density and continuity, we have Green's formula. □

4.3 The embedding theorems and fractional Sobolev spaces

Let $\Omega \in \mathbf{R}^d$. For $m \in \mathbb{Z}_+$, define the Sobolev norms

$$\|u\|_{m,p} := \left(\sum_{0 \le |\alpha| \le m} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and

$$\|u\|_{m,\infty} := \max_{0 \le |\alpha| \le m} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

Then for $m \in \mathbb{Z}_+$ and $1 \le p \le \infty$, the spaces

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \|u\|_{m,p} < \infty\}$$

are Banach spaces.

The fractional Sobolev spaces are then defined as follows. If $s = m + \sigma, \sigma \in [0, 1)$, and with $m \in \mathbb{Z}_+$, the $W^{m+\sigma,p}(\Omega)$ consists of all functions $u \in W^{m,p}(\Omega)$ satisfying

$$\int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+\sigma p}} dx\, dy < \infty, \quad |\alpha| = m.$$

Example 4.14. *Let $\Gamma = (-1, 1)$ and let $u(x) = 0$ for $-1 < x < 0$ and $u(x) = 1$ for $0 < x < 1$. We want to see in which class u belongs to.*

Then certainly $u \in L^2(\Gamma)$ *and*

$$
\begin{aligned}
&\int_\Gamma\int_\Gamma \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{1+2\sigma}}dx\,dy\\
&\quad= \int_{-1}^{1}\int_{-1}^{1} \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{1+2\sigma}}dx\,dy\\
&\quad= 2\int_{-1}^{0}\int_{0}^{1} \frac{1}{(x-y)^{1+2\sigma}}dx\,dy\\
&\quad= 2\int_{-1}^{0}\int_{0}^{1} \frac{1}{(x-y)^{1+2\sigma}}dx\,dy\\
&\quad= \frac{2}{2\sigma}\int_{-1}^{0} \frac{1}{y^{2\sigma}} - \frac{1}{(1-y)^{2\sigma}}\,dy\\
&\quad= \frac{1}{\sigma}\int_{-1}^{0} \frac{1}{(-y)^{2\sigma}} - \frac{1}{(1-y)^{2\sigma}}\,dy\\
&\quad= \frac{1}{\sigma}\int_{-1}^{0} \frac{1}{(-y)^{2\sigma}} - \frac{1}{(1-y)^{2\sigma}}\,dy\\
&\quad= \frac{1}{\sigma}\int_{0}^{1} \frac{1}{(y)^{2\sigma}} - \frac{1}{(1+y)^{2\sigma}}\,dy\\
&\quad= \frac{1}{\sigma(1-2\sigma)}(2-2^{1-2\sigma}) < \infty
\end{aligned}
$$

provided $1-2\sigma>0$. *Therefore* $u \in W^{0,\frac{1}{2}-\epsilon}$ *for all* $\epsilon > 0$.

Theorem 4.16. [Sobolev Embedding Theorem] *Suppose* $\Omega \subset \mathbf{R}^d$ *is an open Lipschitz domain. Then for* $p \in [1,\infty)$ *and integers* $m \ge n$, *set the critical Sobolev index*

$$\gamma = \gamma(m,n,p) = \frac{1}{p} - \frac{m-n}{d}.$$

$$W^{m,p}(\Omega) \subset \begin{cases} W^{n,q}(\Omega) & \text{for } q=\frac{1}{\gamma} \quad \text{if } \gamma>0,\\ W^{n,q}_{loc}(\Omega) & \forall q\in[1,\infty) \quad \text{if } \gamma=0,\\ C^n(\Omega), & \text{if } \gamma<0. \end{cases} \tag{4.12}$$

Moreover, if Ω *is bounded, then*

$$W^{m,p}(\Omega) \subset C^n(\overline{\Omega}) \text{ if } \gamma<0, \tag{4.13}$$

and the following embedding is compact:

$$W^{m,p}(\Omega) \subset W^{n,q'}(\overline{\Omega}) \tag{4.14}$$

for all $q' \in [1,\frac{1}{\gamma})$ *if* $\gamma>0$, *or for all* $q' \in [1,\infty)$ *if* $\gamma = 0$.

Thus if $\Omega \in \mathbf{R}^2$ is an open Lipschitz domain, then

$$H^1(\Omega) \subset L^q_{loc}(\Omega) \quad \forall q\in[1,\infty) \quad \text{since } \gamma = \frac{1}{2}-\frac{1}{2} = 0.$$

and in general

$$H^1(\Omega) \not\subset C^0(\Omega) \quad \text{since } \gamma = \frac{1}{2}-\frac{1}{2} = 0.$$

but

$$W^{1,p}(\Omega) \subset C^0(\Omega) \quad \forall p > 2. \quad \text{since } \gamma = \frac{1}{p} - \frac{1}{2} < 0.$$

If $\Omega \in \mathbf{R}^3$ is an open Lipschitz domain, then

$$H^1(\Omega) \subset L^6(\Omega) \quad \text{since } \gamma = \frac{1}{2} - \frac{2}{3} = \frac{1}{6}.$$

and in general

$$H^1(\Omega) \not\subset C^0(\Omega) \quad \text{since } \gamma = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \quad H^2(\Omega) \subset C^0(\Omega) \quad \text{since } \gamma = \frac{1}{2} - \frac{2}{3} < 0,$$

but

$$W^{1,p}(\Omega) \subset C^0(\Omega) \quad \forall p > 3 \quad \text{since } \gamma = \frac{1}{p} - \frac{1}{3} < 0.$$

Theorem 4.17. *Suppose that Ω is a bounded open subset of $\mathbf{R}^d$ with Lipschitz boundary. Let $m_1, m_2, m \in \mathbb{Z}_+$ be nonnegative integers and $p_1, p_2, p \in [1, \infty)$ be real numbers such that*

$$m_1 \geq m, \quad m_2 \geq m.$$

Suppose that either

1. $\frac{m_1+m_2-m}{d} \geq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}, \quad \frac{m_j-m}{d} > \frac{1}{p_j} - \frac{1}{p}, \quad j = 1, 2;$ *or*
2. $\frac{m_1+m_2-m}{d} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}, \quad \frac{m_j-m}{d} \geq \frac{1}{p_j} - \frac{1}{p}, \quad j = 1, 2.$

Then the multiplication mapping $u, v \to uv$ is continuous bilinear map from $W^{m_1,p_1}(\Omega) \times W^{m_2,p_2}(\Omega)$ into $W^{m,p}(\Omega)$.

Compactness results

Theorem 4.18. *Let Ω be bounded and open in $\mathbf{R}^N$ with a piecewise C^1 boundary Γ. Then the canonical injection $\mathscr{I} : H^1(\Omega) \to L^2(\Omega)$ is compact (i.e. if $(u_j)_j \subset H^1(\Omega)$ with $||u_j||_{1,\Omega} \leq C$ for all j for some constant $C > 0$, then $(u_j)_j$ is relatively compact in $L^2(\Omega)$.)*

Proof. A classical result for Hilbert spaces: Any bounded set of a Hilbert space V is weakly relatively compact. This means if $(f_j)_j \subset V$ is bounded in V, there exists $f \in V$ and a subsequence $(f_{m_j})_j$ such that $f_{m_j} \to f$ weakly in V i.e.,

$$(g, f_{m_j}) \to (g, f) \quad \forall\, g \in V.$$

For the case where Ω is open and 1-regular, there is a continuous linear extension operator $\mathscr{E} : H^1(\Omega) \to H^1(\mathbf{R}^N)$ such that

$$\mathscr{E}u = u \text{ a.e. on } \Omega \quad \forall\, u \in H^1(\Omega)$$

By truncating $\mathscr{E}u$, $\mathscr{E}u$ has a support contained in a fixed compact set K, $K \supset \Omega$. □

Remark 4.4. *If Ω is unbounded, the above theorem may not hold. For instance, let $\Omega = \mathbf{R}^N$. Then*

$$H^1(\mathbf{R}^N) \hookrightarrow L^2(\mathbf{R}^N) \text{ is not compact.}$$

Remark 4.5. *The assumption on the regularity of the domain for the above theorem can be weakened.*

Theorem 4.19. *Let Ω be open, bounded in $\mathbf{R}^N$. Then*

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact.}$$

4.4 Application to elliptic problems and finite element subspace of $H^1(\Omega)$

Let $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k$ such that

1. Ω_k is open in $\mathbf{R}^N$, $\partial\Omega_k$ being piecewise C^1, $k = 1, \cdots, K$.
2. $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.

Theorem 4.20. *Let $u \in C^0(\overline{\Omega})$ such that $u|_{\Omega_i} \in H^1(\Omega_i)$, $i = 1, \cdots, M$. Then $u \in H^1(\Omega)$*

Proof. For each j, define $u_j \in L^2(\Omega)$ by $u_j|_{\Omega_r} = \frac{\partial}{\partial x_j}(u|_{\Omega_k})$, $1 \leq k \leq K$. We want to see $u_j = \frac{\partial u}{\partial x_j}$ in the sense of $\mathscr{D}'(\Omega)$. Then $\frac{\partial u}{\partial x_j} \in L^2(\Omega)$. To show this, for all $\varphi \in \mathscr{D}(\Omega)$,

$$\begin{aligned}
u_j(\varphi) &= \int_\Omega u_j \varphi \, dx = \sum_{k=1}^{K} \int_{\Omega_k} u_j \varphi \, dx = \sum_{k=1}^{K} \int_{\Omega_k} \frac{\partial}{\partial x_j}(u|_{\Omega_k}) \varphi \, dx \\
&= -\sum_{k=1}^{K} \left[\int_{\Omega_k} u|_{\Omega_k} \frac{\partial \varphi}{\partial x_j} \, dx - \int_{\Gamma_k} u \varphi \nu_j \, d\sigma \right] = -\int_\Omega u \frac{\partial \varphi}{\partial x_j} \, dx \\
&= -u(\frac{\partial \varphi}{\partial x_j}) = \frac{\partial u}{\partial x_j}(\varphi).
\end{aligned}$$

Therefore, $u_j = \frac{\partial u}{\partial x_j} \in L^2(\Omega)$, which shows that $u \in H^1(\Omega)$. □

Recall that $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d)$ denotes the unit outward normal to Ω on its boundary Γ.

Theorem 4.21 (Green's Theorem)**.** *Let Ω be a bounded open subset of $\mathbf{R}^d$ with a Lipschitz-continuous boundary Γ.*

1. *If $u, v \in H^1(\Omega)$, then*

$$\int_\Omega u \frac{\partial v}{\partial x_j} \, dx = -\int_\Omega \frac{\partial u}{\partial x_j} v \, dx + \int_\Gamma \gamma_0(uv) \nu_j \, ds. \tag{4.15}$$

2. *If $u, v \in H^1(\Omega)$ and $u \in H^2(\Omega)$, then we have*

$$-\int_\Omega \Delta u v dx = \int_\Omega \nabla u \cdot \nabla v dx - \int_\Gamma \gamma_0 \left(\frac{\partial u}{\partial x_j} v \right) \nu_j d\sigma.$$

Chapter 5

Appendix

5.1 Brief review on mathematical analysis and linear algebra

Unless otherwise stated, by a field $\mathscr{F}$ we mean the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

Definition 5.1. *A vector space $\mathscr{V}$ over a field $\mathscr{F}$ consists of a set of vectors with the two "operations:" (i) the vector addition "+" and (ii) the scalar multiplication such that V is closed under these operations fulfilling the following properties:*

1. *(Closed under "+")* $\forall u, v \in \mathscr{V}, \quad u + v \in \mathscr{V}$
2. *(Closed under the scalar multiplcation)* $\forall \alpha \in \mathscr{F} \quad \forall u \in \mathscr{V}, \quad \alpha u \in \mathscr{V}$
3. *(Existence of zero element, or the identity element for +)* $\exists 0 \in \mathscr{V}$ *such that* $0+u = u = u+0 \quad \forall u \in \mathscr{V}$
4. *(Existence of the inverse element for +)* $\forall u \in \mathscr{V}, \exists -u \in \mathscr{V}$ *such that* $u + (-u) = 0 = (-u) + u$
5. *(The identity element of $\mathscr{F}$)* $1u = u \quad \forall u \in \mathscr{V}$
6. *(Commutative law)* $u + v = v + u \quad \forall u, v \in \mathscr{V}$
7. *(Associative law)* $(u + v) + w = u + (v + w) \quad \forall u, v, w \in \mathscr{V}$
8. *(Associative law)* $(\alpha\beta)u = \alpha(\beta u) \quad \forall \alpha, \beta \in \mathscr{F} \quad \forall u \in \mathscr{V}$
9. *(Distributive law)* $\alpha(u + v) = \alpha u + \alpha v \quad \forall \alpha \in \mathscr{F} \quad \forall u, v \in \mathscr{V}$
10. *(Distributive law)* $(\alpha + \beta)u = \alpha u + \beta u \quad \forall \alpha, \beta \in \mathscr{F} \quad \forall u \in \mathscr{V}$

Definition 5.2. *Let $\mathscr{V}$ be a vector space over $\mathscr{F}$. A nonempty subset $\mathscr{W}$ of $\mathscr{V}$ is called a subspace if the following closure rules hold for $\mathscr{W}$:*

1. *(Closed under "+")* $\forall u, v \in \mathscr{W}, \quad u + v \in \mathscr{W}$
2. *(Closed under the scalar multiplcation)* $\forall \alpha \in \mathscr{F} \quad \forall u \in \mathscr{W}, \quad \alpha u \in \mathscr{W}$.

Example 5.1. *$\mathscr{F}$ is a vector space over itself.*

Example 5.2. *Let d be a positive integer. $\mathscr{F}^d$ is a vector space over $\mathscr{F}$, where* $\mathscr{F}^d = \{\langle v_1, \cdots, v_d\rangle \, | v_j \in \mathscr{F} \quad \forall j = 1, \cdots, d$

Example 5.3. *Let $1 \le p \le \infty$, and $\Omega \subset \mathbb{B}^d$ be an open region. Define*

$$\|f\|_{0,p,\Omega} = \begin{cases} \left[\int_\Omega |f(\mathbf{x})|^p \, d\mathbf{x}\right]^{\frac{1}{p}} & 1 \le p < \infty, \\ ess.sup_{\mathbf{x}\in\Omega} |f(\mathbf{x})| & p = \infty \end{cases}$$

and set

$$L^p(a,b) = \{f : \Omega \mapsto \mathbb{R} \mid \|f\|_{0,p,\Omega} < \infty\}.$$

Then $L^p(a,b)$ *is a vector space over* $\mathbb{R}$. *Indeed, the Minkowski inequality*

$$\|f+g\|_{0,p,\Omega} \le \|f\|_{0,p,\Omega} + \|g\|_{0,p,\Omega} \quad \forall f, g \in L^p(a,b).$$

is the key to verify the closure property under $+$.

Definition 5.3. *Let* $\mathscr{V}$ *and* $\mathscr{W}$ *be vector spaces over* $\mathscr{F}$. *A map* $T : \mathscr{V} \mapsto \mathscr{W}$ *is called linear if the following holds:*

1. *(Preserve the "+" structure)* $\forall u, v \in \mathscr{W}, \quad T(u+v) = T(u) + T(v)$
2. *(Preserve the scalar multiplcation structure)* $\forall \alpha \in \mathscr{F} \quad \forall u \in \mathscr{W}, \quad T(\alpha u) = \alpha T(u)$.

Notational simplification: If T *is a linear map, usually* $T(u)$ *is abbreviated as* Tu.

Definition 5.4. *Let* $\mathscr{V}$ *be a vector space over* $\mathscr{F}$. *An inner product* $(\cdot,\cdot)$ *is a map* $\mathscr{V} \times \mathscr{V} \mapsto \mathscr{F}$ *defined by* $(v,w) \in \mathscr{F}$.

Definition 5.5. *Let* $\mathscr{V}$ *be a vector space over* $\mathscr{F}$. *A norm* $\|\cdot\|_V : V \to \mathbb{R}$ *is a function such that*

1. $\|u\|_V > 0 \quad \forall 0 \ne v \in V$;
2. $\|u+v\|_V \le \|u\|_V + \|v\|_V \quad \forall u, v \in V$;
3. $\|\alpha u\|_V = |\alpha|\, \|u\|_V \quad \forall \alpha \in \mathscr{F} \quad \forall u \in V$.

If a vector space V *over* $\mathscr{F}$ *is equipped with a norm* $\|\cdot\|_V$ *is called a "normed linear space."*

Definition 5.6. *Let* $\mathscr{V}$ *be a vector space over* $\mathscr{F}$. *A seminorm* $|\cdot|_V : V \to \mathbb{R}$ *is a function such that*

1. $|u|_V \ge 0 \quad \forall v \in V$;
2. $|u+v|_V \le \|u\|_V + |v|_V \quad \forall u, v \in V$;
3. $|\alpha u|_V = |\alpha|\, |u|_V \quad \forall \alpha \in \mathscr{F} \quad \forall u \in V$.

A metric space is more general than a normed linear space. Indeed,

Definition 5.7. *Let* M *be a set. A metric* d *is a real–valued function defined on* $M \times M$ *satisfying the following properties:*

1. $d(x,y) \ge 0 \quad \forall x, y \in M$ *(non–negativity);*
2. $d(x,y) = 0 \iff x = y \quad \forall x, y \in M$ *(identity of indiscernibles);*
3. $d(x,y) = d(y,x) \quad \forall x, y \in M$ *(symmetry);*
4. $d(x,z) \le d(x,y) + d(y,z) \quad \forall x, y, z \in M$ *(triangle inequality).*

If a metric d *is equipped, the ordered pair* (M,d) *is called a metric space.*

A normed linear space is a metric space by defining the metric $d(x,y) = c\|x-y\|_V$ for all $x, y \in V$ with any positive constant $c > 0$.

Definition 5.8. *A sequence* $(x_j)_{j=1}^{\infty}$ *in a metric space* (M,d) *is called a Cauchy sequence if for every* $\epsilon > 0$ *there exists an integer* $N = N(\epsilon)$ *such that*

$$d(x_j, x_k) < N(\epsilon) \quad \forall j, k \ge N.$$

Definition 5.9. *A metric space (M, d) is said to be complete if every Cauchy sequence in (M, d) converges to a an element in (M, d).*

Definition 5.10. *A complete normed linear space $(V, \|\cdot\|_V)$ is called a Banach space.*

Definition 5.11. *A complete normed linear space $(V, \|\cdot\|_V)$ with a real or complex innter–product $(\cdot,\cdot)_V$ is called a Hilbert space where the norm is induced from the inner–product, i.e.* $\|v\| = \sqrt{(v, v)} \quad \forall v \in V$.

A Hilbert space V is a generalization of the Euclidian space $\mathbb{R}^d$ or $\mathbb{C}^d$ where V can be a fairly abstract set of elements.

The Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ are Hilbert spaces while $W^{k,p}(\Omega)$ for $p \geq 1$ are Banach spaces with the corresponding Sobolev norms.

Definition 5.12. *Let $(H, \|\cdot\|_H)$ be a Hilbert space and $(V, \|\cdot\|_V)$ be a subspace which is dense and the inclusion $\iota : V \to H$ is continuous, i.e if $\lim_{j\to\infty} \|v_j - v\|_V = 0$ then $\lim_{j\to\infty} \|v_j - v\|_H = 0$. Recall the dual map $\iota^* : V' \to H'$. Now identifying H' with H, the following inclusions and and identifications are called a Gelfand triplet:*

$$V \hookrightarrow H \Longleftrightarrow H' \hookrightarrow V'.$$

The meaning is as follows. If $\phi \in H \Longleftrightarrow H'$, the inclusion implies $\phi \in V'$. Then, for all $v \in V$, one has

$$\langle \phi, v \rangle_{V',V} = \langle \phi, v \rangle_{H',H} = (\phi, v)_H.$$

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