Algebraic Multigrid Methods for the Oseen Problem

Markus Wabro

Joint work with: Walter Zulehner, Linz

Institute of Computational Mathematics
Johannes Kepler University Linz

www.numa.uni-linz.ac.at

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Overview

- Introduction (Navier-Stokes / Oseen equations)
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- Multigrid Methods for the solution of the Oseen equations
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- Multigrid Methods for the solution of the Oseen equations
- Algebraic Multigrid for the Oseen equations
  - Decoupled Approach
  - Coupled Approach
    - Prolongation, AMGe
    - Smoothing
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- Multigrid Methods for the solution of the Oseen equations
- Algebraic Multigrid for the Oseen equations
  - Decoupled Approach
  - Coupled Approach
    - Prolongation, AMGe
    - Smoothing
- Numerical results
Aim

- Complex 3D domain $\Omega$, 

\[
\begin{align*}
\text{Complex 3D domain } \Omega, \\
\text{Finite Element Discretization; } 0@A \quad (u) B T B \quad C \quad 1A @ f g 1A = 0@ f g 1A \\
\text{We want an efficient Solver.}
\end{align*}
\]
Aim

- Complex 3D domain $\Omega$,
- the **incompressible** Navier-Stokes equations
  \[ \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \]
  \[ \text{div } u = 0, \]
  + boundary/initial conditions,
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- the incompressible Navier-Stokes equations
  \[
  \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\
  \text{div } u = 0,
  \]
  + boundary/initial conditions,

- Finite Element Discretization

\[
\begin{pmatrix}
A(u) & B^T \\
B & -C
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix} =
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]
Aim

- Complex 3D domain $\Omega$,
- the incompressible Navier-Stokes equations
  \[ \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \]
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- Finite Element Discretization
  \[ \begin{pmatrix} A(u) & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \]
- We want an efficient Solver.
Convective Term

- **Linearization:** Fixed point iteration $\rightarrow$ Oseen equation

\[-\nu \Delta u + (w \cdot \nabla) u + \nabla p = f\]
Convective Term

- **Linearization:** Fixed point iteration $\leadsto$ Oseen equation

$$-\nu \Delta u + (w \cdot \nabla)u + \nabla p = f$$

- **Stabilization (SUPG):**

$$((w \cdot \nabla)u, v) + \beta_h (w \nabla u, w \nabla v),$$

with $\beta_h = O(h)$. 
Convective Term

- **Linearization**: Fixed point iteration $\mapsto$ **Oseen** equation

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- **Stabilization (SUPG)**:

  $$((w \cdot \nabla)u, v) + \beta_h (w \nabla u, w \nabla v),$$

  with $\beta_h = O(h)$.

  $\rightarrow$ Solution of linear problem?
MGM for Saddle-Point-Problems
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Iterative Decoupling

- Pressure correction methods: SIMPLE, Uzawa
- Precond. Krylov space methods (e.g. GMRES, BiCGstab)

**MG** for the *elliptic* systems.
Iterative Decoupling

- Pressure correction methods: SIMPLE, Uzawa
- Precond. Krylov space methods (e.g. GMRES, BiCGstab)

**MG** for the **elliptic** systems.

**Coupled MG** for the whole system.

Needs appropriate MG components (smoother, inter-grid-transfer).
Decoupled Approach — Examples

\[ u^k + 1 = D^k u^k \]

\[ p^{k + 1} = p^k + \]
Decoupled Approach — Examples

- **SIMPLE:**

\[ \tilde{A}\tilde{u} = f - B^T p_k, \]

\[ \hat{S}\tilde{p} = B\tilde{u} - C p_k - g, \]

\[ u_{k+1} = \tilde{u} - D^{-1} B^T \tilde{p}, \]

\[ p_{k+1} = p_k + \tilde{p}, \]

with \( D \approx A, \) \( \hat{S} = C + B D^{-1} B^T. \)
Decoupled Approach — Examples

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  \[ p_{k+1} = p_k + \tilde{p}, \]

  with \( D \approx A, \hat{S} = C + BD^{-1}B^T. \)

- **Preconditioner of Silvester, Wathen et.al.:**

  \[ P^{-1} = \begin{pmatrix} A_*^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & B^T \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_*^{-1} A_p L_*^{-1} \end{pmatrix}, \]

  \( A_* \approx A, \ Q_* \approx \) pressure mass matrix, \( A_p \ldots \) pressure conv.-diff. operator, \( L_* \approx \) pressure Laplacian (\( \rightarrow \) C. Powell)
Algebraic Multigrid

- Complex geometry $\xrightarrow{\text{discretization}}$ many unknowns.
- No further refinement possible.
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\(\rightarrow\) **Algebraic Multigrid.** Starting from the finest levels generate the coarse levels (almost) only from matrix information.
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- No further refinement possible.

$\rightarrow$ **Algebraic Multigrid.** Starting from the finest levels generate the coarse levels (almost) only from matrix information.

- O.K. for elliptic problems, application to the decoupled approach is straightforward.
- What about coupled AMG?
Coupled Approach — Strategy
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- Separate pressure and velocity-component unknowns.
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- AMG as black box?
Coupled Approach — Strategy

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- Try to translate as much as possible from geometric MG for saddle point problems.
- Try to use as much as possible from elliptic AMG.
- AMG as black box?
Notation

- Grid transfer:
  - Prolongators $P_{i+1}^i = \begin{pmatrix} \tilde{I}_{i+1}^i & J_{i+1}^i \end{pmatrix}$,
  - Restrictors $P_{i+1}^i = P_i^T$.

- Coarse level systems:
  $$K_i = \begin{pmatrix} A_i & B_i^T \\ B_i & -C_i \end{pmatrix} = P_{i-1}^i K_{i-1} P_{i}^{i-1}$$
Notation

- Grid transfer:
  - Prolongators \( P_{i+1}^i = \begin{pmatrix} \tilde{I}_{i+1}^i \\ I_{i+1}^i \\ J_{i+1}^i \end{pmatrix}, \)

  \[
  \tilde{I}_{i+1}^i = \begin{pmatrix}
  I_{i+1}^i \\
  I_{i+1}^i \\
  I_{i+1}^i 
  \end{pmatrix}
  \]

  - Restrictors \( P_{i+1}^i (= P_{i+1}^i)^T \).

- Coarse level systems:
  \[
  K_i = \begin{pmatrix}
  A_i & B_i^T \\
  B_i & -C_i
  \end{pmatrix} = P_{i-1}^i K_{i-1} P_{i-1}^i
  \]

  With two exceptions \((h\text{-dependence in conv. stab.})!\)
Some Mixed Finite Elements

P1-P1-stab:

\[ C = \sum_{\text{elts } K} \alpha h_K^2 (\nabla p, \nabla q)_{0,K} \]
Some Mixed Finite Elements

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\[ C = \alpha \sum_{\text{elts } K} h_K^2 (\nabla p, \nabla q)_{0,K} \]

modified Taylor-Hood:
Some Mixed Finite Elements

P1-P1-stab:

modified Taylor-Hood:

\[ C = \alpha \sum_{\text{elts } K} h^2_K (\nabla p, \nabla q)_{0,K} \]

Crouzeix-Raviart:
Coarsening Strategy

- **P1-P1-stab**: same hierarchy

  ![Coarsening Strategy Diagram]

  - fine mesh
  - 1st level \( u, p \)
  - 2nd level \( u, p \)
  - \( \vdots \)
  - \( N \)-th level \( u, p \)
Coarsening Strategy

- **P1-P1-stab**: same hierarchy

  fine mesh 1st level $\mathbf{u}, p$
  
  ________ 2nd level $\mathbf{u}, p$
  
  ________ ...
  
  ________ $N$-th level $\mathbf{u}, p$

- **mod. Taylor-Hood**: shifted hierarchy (first try)

  fine (\(u\)) mesh 1st level $\mathbf{u}$
  
  coarse (\(p\)) mesh 1st level $p$
  
  ________ 2nd level $\mathbf{u}$
  
  ________ ...
  
  ________ $N$-th level $\mathbf{u}$, $p$
Stability — P1-P1-stab (1)


\[
\sup_{0 \neq \mathbf{v} \in V_h} \frac{(\text{div} \mathbf{v}, p)}{\| \mathbf{v} \|_1} \geq \gamma \| p \|_0 - \delta \left( \sum_{\text{elts } K} h_K^2 \| \nabla p \|_{0,K}^2 \right)^{\frac{1}{2}} \forall p \in Q_h
\]

\[\rightarrow\] Stability
Stability — P1-P1-stab (1)


\[
\sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\text{div} \mathbf{v}, p)}{\| \mathbf{v} \|_1} \geq \gamma \| p \|_0 - \delta \left( \sum_{\text{elts } K} h_K^2 \| \nabla p \|_{0,K}^2 \right)^{\frac{1}{2}} \quad \forall p \in Q_h
\]

\[
\rightarrow \text{Stability}
\]

(In the form \( \sup_{(v,q) \in V \times Q} \frac{\mathcal{B}((w,r),(v,q))}{\|(v,q)\|_X} \geq c \|(w, r)\|_X \),

\[
\mathcal{B}((w, r), (v, q)) = a(w, v) - b(v, r) - b(w, q) - c(r, q)
\]
Stability — P1-P1-stab (2)

AMG case: Assume

- \( A_i \) is of \textit{essentially positive type} i.e. for all \( \mathbf{v}_i \in \mathbf{V}_i \)

\[
\sum_{k,j} (-a_{kj}) (v_k - v_j)^2 \geq \gamma \sum_{k,j} (-a_{kj}^{-}) (v_k - v_j)^2.
\]

- For all \( \mathbf{v}_i \in \mathbf{V}_i \) we can find a \( \prod_{1}^{i+1} \mathbf{v}_i \in \mathbf{V}_{i+1} \) such that

\[
\| \mathbf{v}_i - \tilde{I}_i^{i+1} \prod_{1}^{i+1} \mathbf{v}_i \|_{D_i}^{2} \leq \beta_1 \| \mathbf{v}_i \|_{A_i}^{2}.
\]

(\( \tilde{I}_i^{i+1} \ldots \text{e.g. Ruge, Stüben, '87, '01; Vaněk et al., '95} \))
Stability — P1-P1-stab (2)

AMG case: Assume

- $A_i$ is of essentially positive type i.e. for all $v_i \in V_i$

$$\sum_{k,j} (-a_{kj}) (v_k - v_j)^2 \geq \gamma \sum_{k,j} (-a_{kj}) (v_k - v_j)^2.$$ 

- For all $v_i \in V_i$ we can find a $\Pi^{i+1}_i v_i \in V_{i+1}$ such that

$$\|v_i - \Pi^{i+1}_i v_i\|_{\mathcal{D}_i}^2 \leq \beta_1 \|v_i\|_{A_i}^2.$$

$(\Pi^{i+1}_i \ldots$ e.g. Ruge, Stüben, ’87, ’01; Vaněk et al., ’95)

Then for all levels $i$

$$\sup_{0 \neq v \in V_i} \frac{v B_i^T p}{\|v\|_{A_i}} \geq \gamma \|p\|_{M_i} - \delta (p^T C_i p)^{\frac{1}{2}} \quad \forall p \in Q_i.$$
Stability — mod. Taylor-Hood (1)

arbitrary grid-transfer → problems.
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arbitrary grid-transfer → problems.

- Example: $I_{i+1}^i$ “locally equal” $I \Rightarrow$ unstabilized P1-P1 situation.
Stability — mod. Taylor-Hood (1)

arbitrary grid-transfer → problems.

- Example: $I^i_{i+1}$ "locally equal" $I$ ⇒ unstabilized P1-P1 situation.

- More restrictive coarsening leads to poor approximation.
Stability — mod. Taylor-Hood (1)

arbitrary grid-transfer → problems.

- Example: $I_{i+1}^i$ “locally equal” $I \Rightarrow$ unstabilized P1-P1 situation.

- More restrictive coarsening leads to poor approximation.

- Drop idea of same hierarchy. Guarantee that there are “enough” velocity unknowns. How?
Stability — mod. Taylor-Hood (2)

First try (brute force): double shift.

\[ \text{Refinement} \quad \text{Coarsening} \]

\[ \text{fine (u) mesh} \quad \text{coarse (p) mesh} \]

\[ \begin{array}{c}
1\text{st level} \\
2\text{nd level} \\
N\text{-th level}
\end{array} \]

\[ \begin{array}{c}
u \\
p \\
u \\
p \\
u \\
p
\end{array} \]
Problems

- Stability
- Arbitrary elements?
AMGe

- Stability
- Arbitrary elements?

One possibility

- AMGe (Jones, Vassilevski et.al.):
  - element agglomeration
  - minimal energy basis functions
**AMGe for Crouzeix-Raviart (1)**

Nonconf. geometric
prolongation:

\[
\begin{align*}
\text{Nonconf. geometric} \\
\text{prolongation:}
\end{align*}
\]
AMGe for Crouzeix-Raviart (1)

Nonconf. geometric prolongation:

Conforming AMGe:

\[
\begin{pmatrix}
A_E & C^T \\
C & 0
\end{pmatrix}
\begin{pmatrix}
x_f \\
\lambda
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
z
\end{pmatrix}
\]

\[C = (0 \ I), \ z = \bar{x}_f^b.\]
Nonconforming AMGe:

\[
\begin{pmatrix}
A_E & C^T \\
C & 0
\end{pmatrix}
\begin{pmatrix}
x_f \\
\lambda
\end{pmatrix}
= \begin{pmatrix}
0 \\
z
\end{pmatrix}
\]

\[C, z \text { s.t. } x_f = 1 \text{ on 1-edge, average 0 on 0-edges.}\]
AMGe for Crouzeix-Raviart (2)

Nonconforming AMGe:

\[
\begin{pmatrix}
A_E & C^T \\
C & 0
\end{pmatrix}
\begin{pmatrix}
x_f \\
\lambda
\end{pmatrix} = \begin{pmatrix}
0 \\
z
\end{pmatrix}
\]

\[C, z \text{ s.t. } x_f = 1 \text{ on 1-edge, average 0 on 0-edges.}\]

(in num. practice: truncate small entries to reduce fill-in)
Stability — Crouzeix-Raviart

Lemma: Assume existence of lin.Op. $\Pi^k_{k-1} : U_{k-1} \rightarrow U_k$ with

$$b(\Pi^k_{k-1} v_{k-1}, q_k) = b(v_{k-1}, J^k_{k-1} q_k)$$

$$\forall q_k \in Q_k, v_{k-1} \in U_{k-1}$$

and

$$\|\Pi^k_{k-1} v_{k-1}\|_A \leq C \|v_{k-1}\|_A \quad \forall v_{k-1} \in U_{k-1}.$$  

Then inf-sup in $U_{k-1} \times Q_{k-1}$ implies inf-sup in $U_k \times Q_k$. 
**Lemma:** Assume existence of lin.Op. \( \Pi_{k-1}^k : U_{k-1} \rightarrow U_k \) with

\[
b(\Pi_{k-1}^k v_{k-1}, q_k) = b(v_{k-1}, J_k^{k-1} q_k)
\]

\[
\forall q_k \in Q_k, v_{k-1} \in U_{k-1}
\]

(1)

and

\[
\|\Pi_{k-1}^k v_{k-1}\|_A \leq C\|v_{k-1}\|_A \quad \forall v_{k-1} \in U_{k-1}.
\]

(2)

Then inf-sup in \( U_{k-1} \times Q_{k-1} \) implies inf-sup in \( U_k \times Q_k \).

(1), (2) can be shown using purely geometric arguments.
Smoothing

Use smoothers from GMG for saddle point systems, e.g.

- **Braess-Smoother**:

\[
\hat{A}(\hat{u}^{j+1} - u^j) = f - A u^j - B^T p^j, \\
\hat{S}(p^{j+1} - p^j) = B \hat{u}^{j+1} - C p^j - g, \\
\hat{A}(u^{j+1} - \hat{u}^{j+1}) = -B^T (p^{j+1} - p^j), \\
\hat{S} \approx C + B \hat{A}^{-1} B^T \text{ (AMG)}. 
\]
Smoothing

Use smoothers from GMG for saddle point systems, e.g.

- **Braess-Smoother:**

\[
\hat{A}(\hat{u}^{j+1} - u^j) = f - A u^j - B^T p^j,
\]

\[
\hat{S}(p^{j+1} - p^j) = B \hat{u}^{j+1} - C p^j - g,
\]

\[
\hat{A}(u^{j+1} - \hat{u}^{j+1}) = -B^T (p^{j+1} - p^j),
\]

\[
\hat{S} \approx C + B \hat{A}^{-1} B^T \text{ (AMG)}.
\]

- **Vanka-Smoother:**

  - Solve smaller local problems and combine solutions via (mult.) Schwarz method.
  - Zulehner/Schöberl: theoretical basis for GMG.
Software package AMuSE

All numerical tests were performed using

AMuSE — Algebraic Multigrid for Stokes type Equations,

developed by

M. W. (since 1999).
Numerical Results — P1isoP2-P1

2D valve, \( \approx 10^5 \) vars,
\[ \nu = 0.0005 \]

3D valves, \( \approx 7 \times 10^5 \) vars,
\[ \nu = 0.001 \]
velocity at inlet 0.5,
narrowest part 0.03.
Numerical Results — 3D Valves
Numerical Results — P1-P1-stab

≈ 7 \cdot 10^5 \text{ vars},

mesh: tets, hexas, pyramids, prisms;

\nu = 0.0005
Numerical Results — P1-P1-stab

$\approx 7 \cdot 10^5$ vars,
mesh: tets, hexas, pyramids, prisms;
$\nu = 0.0005$
### Numerical Results — Crouzeix-Raviart

2D valve, W-5-5

<table>
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<th>2</th>
<th>3</th>
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<td>0.15</td>
<td>0.34</td>
<td>0.44</td>
<td>0.49</td>
</tr>
</tbody>
</table>

(at 4th level: 0.8 reduction/minute)
Smoothing — Comparison

- **2D and/or less complex problems:**
  Vanka-smoother leads to a fast and stable solver

- **Complex 3D problems:**
  - Braess-smoother can compensate stability problems.
  - The patches for the Vanka-smoother grow rapidly (for conforming elements) $\rightarrow$ poor efficiency
Summary

- We have a coupled AMG method for saddle point problems originating in the Navier Stokes equations.
- We can use existing components from AMG for elliptic problems, resp. GMG for saddle point problems.
Summary

- We have a coupled AMG method for saddle point problems originating in the Navier Stokes equations.
- We can use existing components from AMG for elliptic problems, resp. GMG for saddle point problems.

BUT

- It is absolutely no Black-Box method!
- We have no rigorous prove for most aspects (e.g. convergence)!